

Chapter 7 – Integrals

Exercise 7.1 Question

1:

sin 2x

Answer

The anti derivative of sin 2x is a function of x whose derivative is sin 2x. It is known that,

$$\frac{d}{dx}(\cos 2x) = -2 \sin 2x$$

$$\Rightarrow \sin 2x = -\frac{1}{2} \frac{d}{dx}(\cos 2x)$$

$$\therefore \sin 2x = \frac{d}{dx} \left(-\frac{1}{2} \cos 2x \right)$$

Therefore, the anti derivative of $\sin 2x$ is $-\frac{1}{2} \cos 2x$

Question 2:

Cos 3x

Answer

The anti derivative of cos 3x is a function of x whose derivative is cos 3x.

It is known that,

$$\frac{d}{dx}(\sin 3x) = 3 \cos 3x$$

$$\Rightarrow \cos 3x = \frac{1}{3} \frac{d}{dx}(\sin 3x)$$

$$\therefore \cos 3x = \frac{d}{dx} \left(\frac{1}{3} \sin 3x \right)$$

Therefore, the anti derivative of $\cos 3x$ is $\frac{1}{3} \sin 3x$.

Question 3:

e^{2x}

Answer

The anti derivative of e^{2x} is the function of x whose derivative is e^{2x} .

It is known that,

$$\frac{d}{dx}(e^{2x}) = 2e^{2x}$$

$$\Rightarrow e^{2x} = \frac{1}{2} \frac{d}{dx}(e^{2x})$$

$$\therefore e^{2x} = \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

Therefore, the anti derivative of e^{2x} is $\frac{1}{2}e^{2x}$.

Question 4:

$(ax+b)^2$

Answer

The anti derivative of $(ax+b)^2$ is the function of x whose derivative is $(ax+b)^2$.

It is known that,

$$\frac{d}{dx}(ax+b)^3 = 3a(ax+b)^2$$

$$\Rightarrow (ax+b)^2 = \frac{1}{3a} \frac{d}{dx}(ax+b)^3$$

$$\therefore (ax+b)^2 = \frac{d}{dx}\left(\frac{1}{3a}(ax+b)^3\right)$$

Therefore, the anti derivative of $(ax+b)^2$ is $\frac{1}{3a}(ax+b)^3$.

Question 5:

$$\sin 2x - 4e^{3x}$$

Answer

The anti derivative of $(\sin 2x - 4e^{3x})$ is the function of x whose derivative is

$$(\sin 2x - 4e^{3x}).$$

It is known that,

$$\frac{d}{dx} \left(-\frac{1}{2} \cos 2x - \frac{4}{3} e^{3x} \right) = \sin 2x - 4e^{3x}$$

Therefore, the anti derivative of $(\sin 2x - 4e^{3x})$ is $\left(-\frac{1}{2} \cos 2x - \frac{4}{3} e^{3x} \right)$.

Question 6:

$$\int (4e^{3x} + 1) dx$$

Answer

$$\int (4e^{3x} + 1) dx$$

$$= 4 \int e^{3x} dx + \int 1 dx$$

$$= 4 \left(\frac{e^{3x}}{3} \right) + x + C$$

$$= \frac{4}{3} e^{3x} + x + C$$

Question 7:

$$\int x^2 \left(1 - \frac{1}{x^2}\right) dx$$

Answer

$$\begin{aligned} & \int x^2 \left(1 - \frac{1}{x^2}\right) dx \\ &= \int (x^2 - 1) dx \\ &= \int x^2 dx - \int 1 dx \\ &= \frac{x^3}{3} - x + C \end{aligned}$$

Question 8:

$$\int (ax^2 + bx + c) dx$$

Answer

$$\begin{aligned} & \int (ax^2 + bx + c) dx \\ &= a \int x^2 dx + b \int x dx + c \int 1 dx \\ &= a \left(\frac{x^3}{3}\right) + b \left(\frac{x^2}{2}\right) + cx + C \\ &= \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C \end{aligned}$$

Question 9:

$$\int (2x^2 + e^x) dx$$

Answer

$$\begin{aligned} & \int (2x^2 + e^x) dx \\ &= 2 \int x^2 dx + \int e^x dx \\ &= 2 \left(\frac{x^3}{3} \right) + e^x + C \\ &= \frac{2}{3} x^3 + e^x + C \end{aligned}$$

Question 10:

$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx$$

Answer

$$\begin{aligned} & \int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx \\ &= \int \left(x + \frac{1}{x} - 2 \right) dx \\ &= \int x dx + \int \frac{1}{x} dx - 2 \int 1 dx \\ &= \frac{x^2}{2} + \log|x| - 2x + C \end{aligned}$$

Question 11:

$$\int \frac{x^3 + 5x^2 - 4}{x^2} dx$$

Answer

$$\begin{aligned} & \int \frac{x^3 + 5x^2 - 4}{x^2} dx \\ &= \int (x + 5 - 4x^{-2}) dx \\ &= \int x dx + 5 \int 1 dx - 4 \int x^{-2} dx \\ &= \frac{x^2}{2} + 5x - 4 \left(\frac{x^{-1}}{-1} \right) + C \\ &= \frac{x^2}{2} + 5x + \frac{4}{x} + C \end{aligned}$$

Question 12:

$$\int \frac{x^3 + 3x + 4}{\sqrt{x}} dx$$

Answer

$$\begin{aligned} & \int \frac{x^3 + 3x + 4}{\sqrt{x}} dx \\ &= \int \left(x^{\frac{5}{2}} + 3x^{\frac{1}{2}} + 4x^{-\frac{1}{2}} \right) dx \\ &= \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{3 \left(x^{\frac{3}{2}} \right)}{\frac{3}{2}} + \frac{4 \left(x^{\frac{1}{2}} \right)}{\frac{1}{2}} + C \\ &= \frac{2}{7} x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8x^{\frac{1}{2}} + C \\ &= \frac{2}{7} x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8\sqrt{x} + C \end{aligned}$$

Question 13:

$$\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$$

Answer

$$\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$$

On dividing, we obtain

$$= \int (x^2 + 1) dx$$

$$= \int x^2 dx + \int 1 dx$$

$$= \frac{x^3}{3} + x + C$$

Question 14:

$$\int (1 - x)\sqrt{x} dx$$

Answer

$$\int (1 - x)\sqrt{x} dx$$

$$= \int \left(\sqrt{x} - x^{\frac{3}{2}} \right) dx$$

$$= \int x^{\frac{1}{2}} dx - \int x^{\frac{3}{2}} dx$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C$$

$$= \frac{2}{3} x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}} + C$$

Question 15:

$$\int \sqrt{x}(3x^2 + 2x + 3) dx$$

Answer

$$\begin{aligned} & \int \sqrt{x}(3x^2 + 2x + 3) dx \\ &= \int \left(3x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 3x^{\frac{1}{2}} \right) dx \\ &= 3 \int x^{\frac{5}{2}} dx + 2 \int x^{\frac{3}{2}} dx + 3 \int x^{\frac{1}{2}} dx \\ &= 3 \left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}} \right) + 2 \left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}} \right) + 3 \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right) + C \\ &= \frac{6}{7} x^{\frac{7}{2}} + \frac{4}{5} x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + C \end{aligned}$$

Question 16:

$$\int (2x - 3 \cos x + e^x) dx$$

Answer

$$\begin{aligned} & \int (2x - 3 \cos x + e^x) dx \\ &= 2 \int x dx - 3 \int \cos x dx + \int e^x dx \\ &= \frac{2x^2}{2} - 3(\sin x) + e^x + C \\ &= x^2 - 3 \sin x + e^x + C \end{aligned}$$

Question 17:

$$\int (2x^2 - 3 \sin x + 5\sqrt{x}) dx$$

Answer

$$\int (2x^2 - 3 \sin x + 5\sqrt{x}) dx$$

$$= 2 \int x^2 dx - 3 \int \sin x dx + 5 \int x^{\frac{1}{2}} dx$$

$$= \frac{2x^3}{3} - 3(-\cos x) + 5 \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right) + C$$

$$= \frac{2}{3} x^3 + 3 \cos x + \frac{10}{3} x^{\frac{3}{2}} + C$$

Question 18:

$$\int \sec x (\sec x + \tan x) dx$$

Answer

$$\int \sec x (\sec x + \tan x) dx$$

$$= \int (\sec^2 x + \sec x \tan x) dx$$

$$= \int \sec^2 x dx + \int \sec x \tan x dx$$

$$= \tan x + \sec x + C$$

Question 19:

$$\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$$

Answer

$$\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$$

$$\begin{aligned}
 &= \int \frac{1}{\frac{\cos^2 x}{\sin^2 x}} dx \\
 &= \int \frac{\sin^2 x}{\cos^2 x} dx \\
 &= \int \tan^2 x dx \\
 &= \int (\sec^2 x - 1) dx \\
 &= \int \sec^2 x dx - \int 1 dx \\
 &= \tan x - x + C
 \end{aligned}$$

Question 20:

$$\int \frac{2 - 3 \sin x}{\cos^2 x} dx$$

Answer

$$\begin{aligned}
 &\int \frac{2 - 3 \sin x}{\cos^2 x} dx \\
 &= \int \left(\frac{2}{\cos^2 x} - \frac{3 \sin x}{\cos^2 x} \right) dx \\
 &= \int 2 \sec^2 x dx - 3 \int \tan x \sec x dx \\
 &= 2 \tan x - 3 \sec x + C
 \end{aligned}$$

Question 21:

The anti derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)$ equals

- (A) $\frac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$ (B) $\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^2 + C$
 (C) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$ (D) $\frac{3}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + C$

Answer

$$\begin{aligned} & \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) dx \\ &= \int x^{\frac{1}{2}} dx + \int x^{-\frac{1}{2}} dx \\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C \\ &= \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C \end{aligned}$$

Hence, the correct Answer is C.

Question 22:

If $\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$, then $f(x)$ is

- (A) $x^4 + \frac{1}{x^3} - \frac{129}{8}$ (B) $x^3 + \frac{1}{x^4} + \frac{129}{8}$
 (C) $x^4 + \frac{1}{x^3} + \frac{129}{8}$ (D) $x^3 + \frac{1}{x^4} - \frac{129}{8}$

Answer

It is given that,

$$\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$$

$$\therefore \text{Anti derivative of } 4x^3 - \frac{3}{x^4} = f(x)$$

$$\begin{aligned}\therefore f(x) &= \int 4x^3 - \frac{3}{x^4} dx \\ f(x) &= 4 \int x^3 dx - 3 \int (x^{-4}) dx \\ f(x) &= 4 \left(\frac{x^4}{4} \right) - 3 \left(\frac{x^{-3}}{-3} \right) + C \\ \therefore f(x) &= x^4 + \frac{1}{x^3} + C\end{aligned}$$

Also,

$$\begin{aligned}f(2) &= 0 \\ \therefore f(2) &= (2)^4 + \frac{1}{(2)^3} + C = 0 \\ \Rightarrow 16 + \frac{1}{8} + C &= 0 \\ \Rightarrow C &= -\left(16 + \frac{1}{8}\right) \\ \Rightarrow C &= \frac{-129}{8} \\ \therefore f(x) &= x^4 + \frac{1}{x^3} - \frac{129}{8}\end{aligned}$$

Hence, the correct Answer is A.

Exercise 7.2

Question 1:

$$\frac{2x}{1+x^2}$$

Answer

Let $1+x^2 = t$

$$,2x \, dx = dt$$

$$\Rightarrow \int \frac{2x}{1+x^2} dx = \int \frac{1}{t} dt$$

$$= \log|t| + C$$

$$= \log|1+x^2| + C$$

$$= \log(1+x^2) + C$$

Question 2:

$$\frac{(\log x)^2}{x}$$

Answer

Let $\log |x| = t$

$$\therefore \frac{1}{x} dx = dt$$

$$\begin{aligned} \Rightarrow \int \frac{(\log |x|)^2}{x} dx &= \int t^2 dt \\ &= \frac{t^3}{3} + C \\ &= \frac{(\log |x|)^3}{3} + C \end{aligned}$$

Question 3:

$$\frac{1}{x + x \log x}$$

Answer

$$\frac{1}{x + x \log x} = \frac{1}{x(1 + \log x)}$$

Let $1 + \log x = t$

$$\therefore \frac{1}{x} dx = dt$$

$$\Rightarrow \int \frac{1}{x(1 + \log x)} dx = \int \frac{1}{t} dt$$

$$= \log |t| + C$$

$$= \log |1 + \log x| + C$$

Question 4: $\sin x$

$$\cdot \sin (\cos x)$$

Answer

$$\sin x \cdot \sin (\cos x)$$

Let $\cos x = t$

$$\therefore -\sin x \, dx = dt$$

$$\begin{aligned}\Rightarrow \int \sin x \cdot \sin(\cos x) \, dx &= - \int \sin t \, dt \\ &= -[-\cos t] + C \\ &= \cos t + C \\ &= \cos(\cos x) + C\end{aligned}$$

Question 5:

$$\sin(ax+b)\cos(ax+b)$$

Answer

$$\sin(ax+b)\cos(ax+b) = \frac{2\sin(ax+b)\cos(ax+b)}{2} = \frac{\sin 2(ax+b)}{2}$$

Let $2(ax+b) = t$

$$\therefore 2adx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{\sin 2(ax+b)}{2} \, dx &= \frac{1}{2} \int \frac{\sin t \, dt}{2a} \\ &= \frac{1}{4a} [-\cos t] + C \\ &= \frac{-1}{4a} \cos 2(ax+b) + C\end{aligned}$$

Question 6:

$$\sqrt{ax+b}$$

Answer

Let $ax + b = t$

$$\Rightarrow adx = dt$$

$$\therefore dx = \frac{1}{a} dt$$

$$\Rightarrow \int (ax+b)^{\frac{1}{2}} dx = \frac{1}{a} \int t^{\frac{1}{2}} dt$$

$$= \frac{1}{a} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + C$$

$$= \frac{2}{3a} (ax+b)^{\frac{3}{2}} + C$$

Question 7:

$$x\sqrt{x+2}$$

Answer

Let $(x+2) = t$

$\therefore dx = dt$

$$\begin{aligned}\Rightarrow \int x\sqrt{x+2} dx &= \int (t-2)\sqrt{t} dt \\ &= \int \left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}} \right) dt \\ &= \int t^{\frac{3}{2}} dt - 2 \int t^{\frac{1}{2}} dt \\ &= \frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 2 \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + C \\ &= \frac{2}{5} t^{\frac{5}{2}} - \frac{4}{3} t^{\frac{3}{2}} + C \\ &= \frac{2}{5} (x+2)^{\frac{5}{2}} - \frac{4}{3} (x+2)^{\frac{3}{2}} + C\end{aligned}$$

Question 8:

$$x\sqrt{1+2x^2}$$

Answer

$$\text{Let } 1 + 2x^2 = t$$

$$\therefore 4x dx = dt$$

$$\begin{aligned}\Rightarrow \int x\sqrt{1+2x^2} dx &= \int \frac{\sqrt{t} dt}{4} \\ &= \frac{1}{4} \int t^{\frac{1}{2}} dt \\ &= \frac{1}{4} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + C \\ &= \frac{1}{6} (1+2x^2)^{\frac{3}{2}} + C\end{aligned}$$

Question 9:

$$(4x+2)\sqrt{x^2+x+1}$$

Answer

Let $x^2+x+1=t$

$$\therefore (2x+1)dx = dt$$

$$\int (4x+2)\sqrt{x^2+x+1} dx$$

$$= \int 2\sqrt{t} dt$$

$$= 2 \int \sqrt{t} dt$$

$$= 2 \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + C$$

$$= \frac{4}{3}(x^2+x+1)^{\frac{3}{2}} + C$$

Question 10:

$$\frac{1}{x-\sqrt{x}}$$

Answer

$$\frac{1}{x-\sqrt{x}} = \frac{1}{\sqrt{x}(\sqrt{x}-1)}$$

Let $(\sqrt{x}-1)=t$

$$\therefore \frac{1}{2\sqrt{x}} dx = dt$$

$$\int \frac{x}{\sqrt{x+4}} dx = \int \frac{(t-4)}{\sqrt{t}} dt \quad \Rightarrow \int \frac{1}{\sqrt{x}(\sqrt{x}-1)} dx = \int \frac{2}{t} dt$$

$$= \int \left(\sqrt{t} - \frac{4}{\sqrt{t}} \right) dt = 2 \log |t| + C$$

$$= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 4 \left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}} \right) + C = 2 \log |\sqrt{x}-1| + C$$

$$= \frac{2}{3} (t)^{\frac{3}{2}} - 8(t)^{\frac{1}{2}} + C$$

$$= \frac{2}{3} t \cdot t^{\frac{1}{2}} - 8t^{\frac{1}{2}} + C$$

$$= \frac{2}{3} t^{\frac{1}{2}} (t-12) + C$$

$$= \frac{2}{3} (x+4)^{\frac{1}{2}} (x+4-12) + C \quad \therefore dx = dt$$

$$= \frac{2}{3} \sqrt{x+4} (x-8) + C$$

Question 11:

$$\frac{x}{\sqrt{x+4}}, x > 0$$

Answer

Let $x+4 = t$

Question 12:

$$(x^3-1)^{\frac{1}{3}} x^5$$

Answer

Let $x^3 - 1 = t$

$$\therefore 3x^2 dx = dt$$

$$\begin{aligned}\Rightarrow \int (x^3 - 1)^{\frac{1}{3}} x^5 dx &= \int (x^3 - 1)^{\frac{1}{3}} x^3 \cdot x^2 dx \\ &= \int t^{\frac{1}{3}} (t+1) \frac{dt}{3} \\ &= \frac{1}{3} \int \left(t^{\frac{4}{3}} + t^{\frac{1}{3}} \right) dt \\ &= \frac{1}{3} \left[\frac{t^{\frac{7}{3}}}{\frac{7}{3}} + \frac{t^{\frac{4}{3}}}{\frac{4}{3}} \right] + C \\ &= \frac{1}{3} \left[\frac{3}{7} t^{\frac{7}{3}} + \frac{3}{4} t^{\frac{4}{3}} \right] + C \\ &= \frac{1}{7} (x^3 - 1)^{\frac{7}{3}} + \frac{1}{4} (x^3 - 1)^{\frac{4}{3}} + C\end{aligned}$$

Question 13:

$$\frac{x^2}{(2+3x^3)^3}$$

Answer

Let $2 + 3x^3 = t$

$\therefore 9x^2 dx = dt$

$$\begin{aligned}\Rightarrow \int \frac{x^2}{(2+3x^3)^3} dx &= \frac{1}{9} \int \frac{dt}{(t)^3} \\ &= \frac{1}{9} \left[\frac{t^{-2}}{-2} \right] + C \\ &= \frac{-1}{18} \left(\frac{1}{t^2} \right) + C \\ &= \frac{-1}{18(2+3x^3)^2} + C\end{aligned}$$

Question 14:

$$\frac{1}{x(\log x)^m}, x > 0$$

Answer

Let $\log x = t$

$$\begin{aligned}\frac{1}{x} dx &= dt \\ \therefore x\end{aligned}$$

$$\begin{aligned}\Rightarrow \int \frac{1}{x(\log x)^m} dx &= \int \frac{dt}{(t)^m} \\ &= \left(\frac{t^{-m+1}}{1-m} \right) + C \\ &= \frac{(\log x)^{1-m}}{(1-m)} + C\end{aligned}$$

Question 15:

$$\frac{x}{9-4x^2}$$

Answer

Let $9-4x^2 = t$

$\therefore -8x \, dx = dt$

$$\begin{aligned}\Rightarrow \int \frac{x}{9-4x^2} dx &= \frac{-1}{8} \int \frac{1}{t} dt \\ &= \frac{-1}{8} \log|t| + C \\ &= \frac{-1}{8} \log|9-4x^2| + C\end{aligned}$$

Question 16:

$$e^{2x+3}$$

Answer

Let $2x+3 = t$

$\therefore 2dx = dt$

$$\begin{aligned}\Rightarrow \int e^{2x+3} dx &= \frac{1}{2} \int e^t dt \\ &= \frac{1}{2} (e^t) + C \\ &= \frac{1}{2} e^{(2x+3)} + C\end{aligned}$$

Question 17:

$$\frac{x}{e^{x^2}}$$

Answer

Let $x^2 = t$

$$\therefore 2x dx = dt$$

$$\Rightarrow \int \frac{x}{e^{x^2}} dx = \frac{1}{2} \int \frac{1}{e^t} dt$$

$$= \frac{1}{2} \int e^{-t} dt$$

$$= \frac{1}{2} \left(\frac{e^{-t}}{-1} \right) + C$$

$$= -\frac{1}{2} e^{-x^2} + C$$

$$= \frac{-1}{2e^{x^2}} + C$$

Question 18:

$$\frac{e^{\tan^{-1} x}}{1+x^2}$$

Answer

Let $\tan^{-1} x = t$

$$\therefore \frac{1}{1+x^2} dx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{e^{\tan^{-1}x}}{1+x^2} dx &= \int e^t dt \\ &= e^t + C \\ &= e^{\tan^{-1}x} + C\end{aligned}$$

Question 19:

$$\frac{e^{2x}-1}{e^{2x}+1}$$

Answer

$$\frac{e^{2x}-1}{e^{2x}+1}$$

Dividing numerator and denominator by e^x , we obtain

$$\frac{\frac{(e^{2x}-1)}{e^x}}{\frac{(e^{2x}+1)}{e^x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\text{Let } e^x + e^{-x} = t$$

$$\therefore (e^x - e^{-x}) dx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{e^{2x}-1}{e^{2x}+1} dx &= \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx \\ &= \int \frac{dt}{t} \\ &= \log|t| + C \\ &= \log|e^x + e^{-x}| + C\end{aligned}$$

Question 20:

$$\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$$

Answer

Let $e^{2x} + e^{-2x} = t$

$$\therefore (2e^{2x} - 2e^{-2x}) dx = dt$$

$$\Rightarrow 2(e^{2x} - e^{-2x}) dx = dt$$

$$\begin{aligned} \Rightarrow \int \left(\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \right) dx &= \int \frac{dt}{2t} \\ &= \frac{1}{2} \int \frac{1}{t} dt \\ &= \frac{1}{2} \log|t| + C \\ &= \frac{1}{2} \log|e^{2x} + e^{-2x}| + C \end{aligned}$$

Question 21:

$$\tan^2(2x-3)$$

Answer

$$\tan^2(2x-3) = \sec^2(2x-3) - 1$$

Let $2x - 3 = t$

$$\therefore 2dx = dt$$

$$\begin{aligned}\Rightarrow \int \tan^2(2x-3) dx &= \int [(\sec^2(2x-3)) - 1] dx \\ &= \frac{1}{2} \int (\sec^2 t) dt - \int 1 dx \\ &= \frac{1}{2} \int \sec^2 t dt - \int 1 dx \\ &= \frac{1}{2} \tan t - x + C \\ &= \frac{1}{2} \tan(2x-3) - x + C\end{aligned}$$

Question 22:

$$\sec^2(7-4x)$$

Answer

$$\text{Let } 7 - 4x = t$$

$$\therefore -4dx = dt$$

$$\begin{aligned}\therefore \int \sec^2(7-4x) dx &= \frac{-1}{4} \int \sec^2 t dt \\ &= \frac{-1}{4} (\tan t) + C \\ &= \frac{-1}{4} \tan(7-4x) + C\end{aligned}$$

Question 23:

$$\frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

Answer

Let $\sin^{-1} x = t$

$$\therefore \frac{1}{\sqrt{1-x^2}} dx = dt$$

$$\Rightarrow \int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int t dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{(\sin^{-1} x)^2}{2} + C$$

Question 24:

$$\frac{2 \cos x - 3 \sin x}{6 \cos x + 4 \sin x}$$

Answer

$$\frac{2 \cos x - 3 \sin x}{6 \cos x + 4 \sin x} = \frac{2 \cos x - 3 \sin x}{2(3 \cos x + 2 \sin x)}$$

Let $3 \cos x + 2 \sin x = t$

$$\therefore (-3 \sin x + 2 \cos x) dx = dt$$

$$\begin{aligned}\int \frac{2 \cos x - 3 \sin x}{6 \cos x + 4 \sin x} dx &= \int \frac{dt}{2t} \\ &= \frac{1}{2} \int \frac{1}{t} dt \\ &= \frac{1}{2} \log |t| + C \\ &= \frac{1}{2} \log |2 \sin x + 3 \cos x| + C\end{aligned}$$

Question 25:

$$\frac{1}{\cos^2 x (1 + \tan x)^2} = \frac{\sec^2 x}{(1 + \tan x)^2}$$

Let $\sqrt{x} = t$
 Let $(1 + \tan x) = t$
 $\frac{1}{2\sqrt{x}} dx = dt$
 $\therefore -\sec^2 x dx = dt$

$$\begin{aligned} \Rightarrow \int \frac{\sec^2 x}{(1 + \tan x)^2} dx &= \int \frac{-dt}{t^2} \\ &= -\int t^{-2} dt \\ &= +\frac{1}{t} + C \\ &= \frac{1}{(1 + \tan x)} + C \\ \Rightarrow \int \frac{\cos \sqrt{x}}{\sqrt{x}} dx &= 2 \int \cos t dt \\ &= 2 \sin t + C \\ &= 2 \sin \sqrt{x} + C \end{aligned}$$

Question 27:

$$\int \sqrt{\sin 2x} \cos 2x dx$$

Answer

Let $\sin 2x = t$

$\therefore 2 \cos 2x dx = dt$

$$\begin{aligned} \Rightarrow \int \sqrt{\sin 2x} \cos 2x dx &= \frac{1}{2} \int \sqrt{t} dt \\ &= \frac{1}{2} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + C \\ &= \frac{1}{3} t^{\frac{3}{2}} + C \\ &= \frac{1}{3} (\sin 2x)^{\frac{3}{2}} + C \end{aligned}$$

Question 28:

$$\frac{\cos x}{\sqrt{1 + \sin x}}$$

Answer

Let $1 + \sin x = t$

$\therefore \cos x \, dx = dt$

$$\begin{aligned}\Rightarrow \int \frac{\cos x}{\sqrt{1 + \sin x}} \, dx &= \int \frac{dt}{\sqrt{t}} \\ &= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C \\ &= 2\sqrt{t} + C \\ &= 2\sqrt{1 + \sin x} + C\end{aligned}$$

Question 29: cot

x log sin x

Answer

Let $\log \sin x = t$

$$\Rightarrow \frac{1}{\sin x} \cdot \cos x \, dx = dt$$

$\therefore \cot x \, dx = dt$

$$\begin{aligned}\Rightarrow \int \cot x \log \sin x \, dx &= \int t \, dt \\ &= \frac{t^2}{2} + C \\ &= \frac{1}{2}(\log \sin x)^2 + C\end{aligned}$$

Question 30:

$$\frac{\sin x}{1 + \cos x}$$

Answer

$$\text{Let } 1 + \cos x = t$$

$$\therefore -\sin x \, dx = dt$$

$$\begin{aligned} \Rightarrow \int \frac{\sin x}{1 + \cos x} \, dx &= \int -\frac{dt}{t} \\ &= -\log|t| + C \\ &= -\log|1 + \cos x| + C \end{aligned}$$

Question 31:

$$\frac{\sin x}{(1 + \cos x)^2}$$

Answer

$$\text{Let } 1 + \cos x = t$$

$$\therefore -\sin x \, dx = dt$$

$$\begin{aligned} \Rightarrow \int \frac{\sin x}{(1 + \cos x)^2} dx &= \int -\frac{dt}{t^2} \\ &= -\int t^{-2} dt \\ &= \frac{1}{t} + C \\ &= \frac{1}{1 + \cos x} + C \end{aligned}$$

Question 32:

$$\frac{1}{1 + \cot x}$$

Answer

$$\begin{aligned} \text{Let } I &= \int \frac{1}{1 + \cot x} dx \\ &= \int \frac{1}{1 + \frac{\cos x}{\sin x}} dx \\ &= \int \frac{\sin x}{\sin x + \cos x} dx \\ &= \frac{1}{2} \int \frac{2 \sin x}{\sin x + \cos x} dx \\ &= \frac{1}{2} \int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{(\sin x + \cos x)} dx \\ &= \frac{1}{2} \int 1 dx + \frac{1}{2} \int \frac{\sin x - \cos x}{\sin x + \cos x} dx \\ &= \frac{1}{2}(x) + \frac{1}{2} \int \frac{\sin x - \cos x}{\sin x + \cos x} dx \end{aligned}$$

Let $\sin x + \cos x = t \Rightarrow (\cos x - \sin x) dx = dt$

$$\begin{aligned} \therefore I &= \frac{x}{2} + \frac{1}{2} \int \frac{-(dt)}{t} \\ &= \frac{x}{2} - \frac{1}{2} \log|t| + C \\ &= \frac{x}{2} - \frac{1}{2} \log|\sin x + \cos x| + C \end{aligned}$$

Question 33:

$$\frac{1}{1 - \tan x}$$

Answer

$$\begin{aligned} \text{Let } I &= \int \frac{1}{1 - \tan x} dx \\ &= \int \frac{1}{1 - \frac{\sin x}{\cos x}} dx \\ &= \int \frac{\cos x}{\cos x - \sin x} dx \\ &= \frac{1}{2} \int \frac{2 \cos x}{\cos x - \sin x} dx \\ &= \frac{1}{2} \int \frac{(\cos x - \sin x) + (\cos x + \sin x)}{(\cos x - \sin x)} dx \\ &= \frac{1}{2} \int 1 dx + \frac{1}{2} \int \frac{\cos x + \sin x}{\cos x - \sin x} dx \\ &= \frac{x}{2} + \frac{1}{2} \int \frac{\cos x + \sin x}{\cos x - \sin x} dx \end{aligned}$$

Put $\cos x - \sin x = t \Rightarrow (-\sin x - \cos x) dx = dt$

$$\begin{aligned} \therefore I &= \frac{x}{2} + \frac{1}{2} \int \frac{-dt}{t} \\ &= \frac{x}{2} - \frac{1}{2} \log|t| + C \\ &= \frac{x}{2} - \frac{1}{2} \log|\cos x - \sin x| + C \end{aligned}$$

Question 34:

$$\frac{\sqrt{\tan x}}{\sin x \cos x}$$

Answer

$$\begin{aligned} \text{Let } I &= \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx \\ &= \int \frac{\sqrt{\tan x} \times \cos x}{\sin x \cos x \times \cos x} dx \\ &= \int \frac{\sqrt{\tan x}}{\tan x \cos^2 x} dx \\ &= \int \frac{\sec^2 x dx}{\sqrt{\tan x}} \end{aligned}$$

$$\text{Let } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{dt}{\sqrt{t}} \\ &= 2\sqrt{t} + C \\ &= 2\sqrt{\tan x} + C \end{aligned}$$

Question 35:

$$\frac{(1 + \log x)^2}{x}$$

Answer

$$\text{Let } 1 + \log x = t$$

$$\therefore \frac{1}{x} dx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{(1 + \log x)^2}{x} dx &= \int t^2 dt \\ &= \frac{t^3}{3} + C \\ &= \frac{(1 + \log x)^3}{3} + C\end{aligned}$$

Question 36:

$$\frac{(x+1)(x+\log x)^2}{x}$$

Answer

$$\frac{(x+1)(x+\log x)^2}{x} = \left(\frac{x+1}{x}\right)(x+\log x)^2 = \left(1 + \frac{1}{x}\right)(x+\log x)^2$$

Let $(x + \log x) = t$

$$\therefore \left(1 + \frac{1}{x}\right) dx = dt$$

$$\begin{aligned}\Rightarrow \int \left(1 + \frac{1}{x}\right)(x + \log x)^2 dx &= \int t^2 dt \\ &= \frac{t^3}{3} + C \\ &= \frac{1}{3}(x + \log x)^3 + C\end{aligned}$$

Question 37:

$$\frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8}$$

Answer

Let $x^4 = t$

$$\therefore 4x^3 dx = dt$$

$$\Rightarrow \int \frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8} dx = \frac{1}{4} \int \frac{\sin(\tan^{-1} t)}{1+t^2} dt \quad \dots(1)$$

Let $\tan^{-1} t = u$

$$\therefore \frac{1}{1+t^2} dt = du$$

From (1), we obtain

$$\begin{aligned} \int \frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8} dx &= \frac{1}{4} \int \sin u du \\ &= \frac{1}{4} (-\cos u) + C \end{aligned}$$

$$= \frac{-1}{4} \cos(\tan^{-1} t) + C$$

$$= \frac{-1}{4} \cos(\tan^{-1} x^4) + C$$

Question 38:

$$\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx$$

equals

- (A) $10^x - x^{10} + C$ (B) $10^x + x^{10} + C$
(C) $(10^x - x^{10})^{-1} + C$ (D) $\log(10^x + x^{10}) + C$

Answer

Let $x^{10} + 10^x = t$

$$\begin{aligned} & (10x^9 + 10^x \log_e 10) dx = dt \\ \therefore & \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx = \int \frac{dt}{t} \\ & = \log t + C \\ & = \log(10^x + x^{10}) + C \end{aligned}$$

Hence, the correct Answer is D.

Question 39:

$\int \frac{dx}{\sin^2 x \cos^2 x}$ equals

- A. $\tan x + \cot x + C$
- B. $\tan x - \cot x + C$
- C. $\tan x \cot x + C$
- D. $\tan x - \cot 2x + C$

Answer

$$\begin{aligned}\text{Let } I &= \int \frac{dx}{\sin^2 x \cos^2 x} \\ &= \int \frac{1}{\sin^2 x \cos^2 x} dx \\ &= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx \\ &= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx \\ &= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx \\ &= \tan x - \cot x + C\end{aligned}$$

Hence, the correct Answer is B.

Exercise 7.3

Question 1:

$$\sin^2(2x+5)$$

Answer

$$\begin{aligned}\sin^2(2x+5) &= \frac{1-\cos 2(2x+5)}{2} = \frac{1-\cos(4x+10)}{2} \\ \Rightarrow \int \sin^2(2x+5) dx &= \int \frac{1-\cos(4x+10)}{2} dx \\ &= \frac{1}{2} \int 1 dx - \frac{1}{2} \int \cos(4x+10) dx \\ &= \frac{1}{2}x - \frac{1}{2} \left(\frac{\sin(4x+10)}{4} \right) + C \\ &= \frac{1}{2}x - \frac{1}{8} \sin(4x+10) + C\end{aligned}$$

Question 2:

$$\sin 3x \cos 4x$$

Answer

$$\sin A \cos B = \frac{1}{2} \{ \sin(A+B) + \sin(A-B) \}$$

It is known that,

$$\begin{aligned}\therefore \int \sin 3x \cos 4x dx &= \frac{1}{2} \int \{ \sin(3x+4x) + \sin(3x-4x) \} dx \\ &= \frac{1}{2} \int \{ \sin 7x + \sin(-x) \} dx \\ &= \frac{1}{2} \int \{ \sin 7x - \sin x \} dx \\ &= \frac{1}{2} \int \sin 7x dx - \frac{1}{2} \int \sin x dx \\ &= \frac{1}{2} \left(\frac{-\cos 7x}{7} \right) - \frac{1}{2} (-\cos x) + C \\ &= \frac{-\cos 7x}{14} + \frac{\cos x}{2} + C\end{aligned}$$

Question 3: $\cos 2x$
 $\cos 4x \cos 6x$

Answer

$$\cos A \cos B = \frac{1}{2} \{ \cos(A+B) + \cos(A-B) \}$$

It is known that,

$$\begin{aligned} \therefore \int \cos 2x (\cos 4x \cos 6x) dx &= \int \cos 2x \left[\frac{1}{2} \{ \cos(4x+6x) + \cos(4x-6x) \} \right] dx \\ &= \frac{1}{2} \int \{ \cos 2x \cos 10x + \cos 2x \cos(-2x) \} dx \\ &= \frac{1}{2} \int \{ \cos 2x \cos 10x + \cos^2 2x \} dx \\ &= \frac{1}{2} \int \left[\frac{1}{2} \cos(2x+10x) + \cos(2x-10x) \right] + \left(\frac{1+\cos 4x}{2} \right) dx \\ &= \frac{1}{4} \int (\cos 12x + \cos 8x + 1 + \cos 4x) dx \\ &= \frac{1}{4} \left[\frac{\sin 12x}{12} + \frac{\sin 8x}{8} + x + \frac{\sin 4x}{4} \right] + C \end{aligned}$$

Question 4: \sin^3

$(2x + 1)$

Answer

$$I = \int \sin^3(2x+1)$$

Let

$$\begin{aligned} \Rightarrow \int \sin^3(2x+1) dx &= \int \sin^2(2x+1) \cdot \sin(2x+1) dx \\ &= \int (1 - \cos^2(2x+1)) \sin(2x+1) dx \end{aligned}$$

$$\text{Let } \cos(2x+1) = t$$

$$\Rightarrow -2 \sin(2x+1) dx = dt$$

$$\Rightarrow \sin(2x+1) dx = \frac{-dt}{2}$$

$$\begin{aligned}
 \Rightarrow I &= \frac{-1}{2} \int (1-t^2) dt \\
 &= \frac{-1}{2} \left\{ t - \frac{t^3}{3} \right\} \\
 &= \frac{-1}{2} \left\{ \cos(2x+1) - \frac{\cos^3(2x+1)}{3} \right\} \\
 &= \frac{-\cos(2x+1)}{2} + \frac{\cos^3(2x+1)}{6} + C
 \end{aligned}$$

Question 5: $\sin^3 x$

$x \cos^3 x$

Answer

$$\begin{aligned}
 \text{Let } I &= \int \sin^3 x \cos^3 x \cdot dx \\
 &= \int \cos^3 x \cdot \sin^2 x \cdot \sin x \cdot dx \\
 &= \int \cos^3 x (1 - \cos^2 x) \sin x \cdot dx
 \end{aligned}$$

Let $\cos x = t$

$$\Rightarrow -\sin x \cdot dx = dt$$

$$\begin{aligned}
 \Rightarrow I &= - \int t^3 (1-t^2) dt \\
 &= - \int (t^3 - t^5) dt \\
 &= - \left\{ \frac{t^4}{4} - \frac{t^6}{6} \right\} + C \\
 &= - \left\{ \frac{\cos^4 x}{4} - \frac{\cos^6 x}{6} \right\} + C \\
 &= \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C
 \end{aligned}$$

Question 6:

$\sin x \sin 2x \sin 3x$

Answer

$$\sin A \sin B = \frac{1}{2} \{ \cos(A - B) - \cos(A + B) \}$$

It is known that,

$$\begin{aligned} \therefore \int \sin x \sin 2x \sin 3x \, dx &= \int \left[\sin x \cdot \frac{1}{2} \{ \cos(2x - 3x) - \cos(2x + 3x) \} \right] dx \\ &= \frac{1}{2} \int (\sin x \cos(-x) - \sin x \cos 5x) \, dx \\ &= \frac{1}{2} \int (\sin x \cos x - \sin x \cos 5x) \, dx \\ &= \frac{1}{2} \int \frac{\sin 2x}{2} \, dx - \frac{1}{2} \int \sin x \cos 5x \, dx \\ &= \frac{1}{4} \left[\frac{-\cos 2x}{2} \right] - \frac{1}{2} \int \left\{ \frac{1}{2} \sin(x + 5x) + \sin(x - 5x) \right\} dx \\ &= \frac{-\cos 2x}{8} - \frac{1}{4} \int (\sin 6x + \sin(-4x)) \, dx \\ &= \frac{-\cos 2x}{8} - \frac{1}{4} \left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{4} \right] + C \\ &= \frac{-\cos 2x}{8} - \frac{1}{8} \left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{2} \right] + C \\ &= \frac{1}{8} \left[\frac{\cos 6x}{3} - \frac{\cos 4x}{2} - \cos 2x \right] + C \end{aligned}$$

Question 7: \sin

$4x \sin 8x$

Answer

$$\sin A \sin B = \frac{1}{2} \cos(A - B) - \cos(A + B)$$

It is known that,

$$\begin{aligned} \therefore \int \sin 4x \sin 8x \, dx &= \int \left\{ \frac{1}{2} \cos(4x - 8x) - \cos(4x + 8x) \right\} dx \\ &= \frac{1}{2} \int (\cos(-4x) - \cos 12x) \, dx \\ &= \frac{1}{2} \int (\cos 4x - \cos 12x) \, dx \\ &= \frac{1}{2} \left[\frac{\sin 4x}{4} - \frac{\sin 12x}{12} \right] \end{aligned}$$

Question 8:

$$\frac{1 - \cos x}{1 + \cos x}$$

Answer

$$\frac{1 - \cos x}{1 + \cos x} = \frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \quad \left[2 \sin^2 \frac{x}{2} = 1 - \cos x \text{ and } 2 \cos^2 \frac{x}{2} = 1 + \cos x \right]$$

$$= \tan^2 \frac{x}{2}$$

$$= \left(\sec^2 \frac{x}{2} - 1 \right)$$

$$\therefore \int \frac{1 - \cos x}{1 + \cos x} dx = \int \left(\sec^2 \frac{x}{2} - 1 \right) dx$$

$$= \left[\frac{\tan \frac{x}{2}}{\frac{1}{2}} - x \right] + C$$

$$= 2 \tan \frac{x}{2} - x + C$$

Question 9:

$$\frac{\cos x}{1 + \cos x}$$

Answer

$$\begin{aligned}\frac{\cos x}{1 + \cos x} &= \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} && \left[\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \text{ and } \cos x = 2 \cos^2 \frac{x}{2} - 1 \right] \\ &= \frac{1}{2} \left[1 - \tan^2 \frac{x}{2} \right] \\ \therefore \int \frac{\cos x}{1 + \cos x} dx &= \frac{1}{2} \int \left(1 - \tan^2 \frac{x}{2} \right) dx \\ &= \frac{1}{2} \int \left(1 - \sec^2 \frac{x}{2} + 1 \right) dx \\ &= \frac{1}{2} \int \left(2 - \sec^2 \frac{x}{2} \right) dx \\ &= \frac{1}{2} \left[2x - \frac{\tan \frac{x}{2}}{\frac{1}{2}} \right] + C \\ &= x - \tan \frac{x}{2} + C\end{aligned}$$

Question 10:

$\sin^4 x$

Answer

$$\begin{aligned} \cos^4 2x &= (\cos^2 2x)^2 \\ &= \left(\frac{1+\cos 4x}{2}\right)^2 \\ &= \frac{1}{4} [1 + \cos^2 4x + 2 \cos 4x] \\ &= \frac{1}{4} \left[1 + \left(\frac{1+\cos 8x}{2}\right) + 2 \cos 4x\right] \\ &= \frac{1}{4} \left[1 + \frac{1}{2} + \frac{\cos 8x}{2} + 2 \cos 4x\right] \\ &= \frac{1}{4} \left[\frac{3}{2} + \frac{\cos 8x}{2} + 2 \cos 4x\right] \\ \therefore \int \cos^4 2x \, dx &= \int \left(\frac{3}{8} + \frac{\cos 8x}{8} + \frac{\cos 4x}{2}\right) dx \\ &= \frac{3}{8}x + \frac{\sin 8x}{64} + \frac{\sin 4x}{8} + C \end{aligned}$$

Question 11:

$\cos^4 2x$

Answer

$$\begin{aligned} \sin^4 x &= \sin^2 x \sin^2 x \\ &= \left(\frac{1-\cos 2x}{2}\right) \left(\frac{1-\cos 2x}{2}\right) \\ &= \frac{1}{4} (1-\cos 2x)^2 \\ &= \frac{1}{4} [1 + \cos^2 2x - 2 \cos 2x] \\ &= \frac{1}{4} \left[1 + \left(\frac{1+\cos 4x}{2}\right) - 2 \cos 2x\right] \\ &= \frac{1}{4} \left[1 + \frac{1}{2} + \frac{1}{2} \cos 4x - 2 \cos 2x\right] \\ &= \frac{1}{4} \left[\frac{3}{2} + \frac{1}{2} \cos 4x - 2 \cos 2x\right] \\ \therefore \int \sin^4 x \, dx &= \frac{1}{4} \int \left[\frac{3}{2} + \frac{1}{2} \cos 4x - 2 \cos 2x\right] dx \\ &= \frac{1}{4} \left[\frac{3}{2}x + \frac{1}{2} \left(\frac{\sin 4x}{4}\right) - \frac{2 \sin 2x}{2}\right] + C \\ &= \frac{1}{8} \left[3x + \frac{\sin 4x}{4} - 2 \sin 2x\right] + C \\ &= \frac{3x}{8} - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C \end{aligned}$$

Question 12:

$$\frac{\sin^2 x}{1 + \cos x}$$

Answer

$$\begin{aligned} \frac{\sin^2 x}{1 + \cos x} &= \frac{\left(2 \sin \frac{x}{2} \cos \frac{x}{2}\right)^2}{2 \cos^2 \frac{x}{2}} \quad \left[\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}; \cos x = 2 \cos^2 \frac{x}{2} - 1 \right] \\ &= \frac{4 \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \\ &= 2 \sin^2 \frac{x}{2} \\ &= 1 - \cos x \\ \therefore \int \frac{\sin^2 x}{1 + \cos x} dx &= \int (1 - \cos x) dx \\ &= x - \sin x + C \end{aligned}$$

Question 13:

$$\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$$

Answer

$$\begin{aligned} \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} &= \frac{-2 \sin \frac{2x+2\alpha}{2} \sin \frac{2x-2\alpha}{2}}{-2 \sin \frac{x+\alpha}{2} \sin \frac{x-\alpha}{2}} \quad \left[\cos C - \cos D = -2 \sin \frac{C+D}{2} \sin \frac{C-D}{2} \right] \\ &= \frac{\sin(x+\alpha) \sin(x-\alpha)}{\sin\left(\frac{x+\alpha}{2}\right) \sin\left(\frac{x-\alpha}{2}\right)} \\ &= \frac{\left[2 \sin\left(\frac{x+\alpha}{2}\right) \cos\left(\frac{x+\alpha}{2}\right) \right] \left[2 \sin\left(\frac{x-\alpha}{2}\right) \cos\left(\frac{x-\alpha}{2}\right) \right]}{\sin\left(\frac{x+\alpha}{2}\right) \sin\left(\frac{x-\alpha}{2}\right)} \\ &= 4 \cos\left(\frac{x+\alpha}{2}\right) \cos\left(\frac{x-\alpha}{2}\right) \\ &= 2 \left[\cos\left(\frac{x+\alpha}{2} + \frac{x-\alpha}{2}\right) + \cos\left(\frac{x+\alpha}{2} - \frac{x-\alpha}{2}\right) \right] \\ &= 2 [\cos(x) + \cos \alpha] \\ &= 2 \cos x + 2 \cos \alpha \\ \therefore \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx &= \int 2 \cos x + 2 \cos \alpha \\ &= 2 [\sin x + x \cos \alpha] + C \end{aligned}$$

Question 14:

$$\frac{\cos x - \sin x}{1 + \sin 2x}$$

Answer

$$\frac{\cos x - \sin x}{1 + \sin 2x} = \frac{\cos x - \sin x}{(\sin^2 x + \cos^2 x) + 2 \sin x \cos x}$$

$[\sin^2 x + \cos^2 x = 1; \sin 2x = 2 \sin x \cos x]$

$$= \frac{\cos x - \sin x}{(\sin x + \cos x)^2}$$

Let $\sin x + \cos x = t$

$$\therefore (\cos x - \sin x) dx = dt$$

$$\begin{aligned} \Rightarrow \int \frac{\cos x - \sin x}{1 + \sin 2x} dx &= \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2} dx \\ &= \int \frac{dt}{t^2} \\ &= \int t^{-2} dt \\ &= -t^{-1} + C \\ &= -\frac{1}{t} + C \\ &= \frac{-1}{\sin x + \cos x} + C \end{aligned}$$

Question 15:

$$\tan^3 2x \sec 2x$$

Answer

$$\begin{aligned}\tan^3 2x \sec 2x &= \tan^2 2x \tan 2x \sec 2x \\ &= (\sec^2 2x - 1) \tan 2x \sec 2x \\ &= \sec^2 2x \cdot \tan 2x \sec 2x - \tan 2x \sec 2x \\ \therefore \int \tan^3 2x \sec 2x \, dx &= \int \sec^2 2x \tan 2x \sec 2x \, dx - \int \tan 2x \sec 2x \, dx \\ &= \int \sec^2 2x \tan 2x \sec 2x \, dx - \frac{\sec 2x}{2} + C\end{aligned}$$

Let $\sec 2x = t$

$$\therefore 2 \sec 2x \tan 2x \, dx = dt$$

$$\begin{aligned}\therefore \int \tan^3 2x \sec 2x \, dx &= \frac{1}{2} \int t^2 dt - \frac{\sec 2x}{2} + C \\ &= \frac{t^3}{6} - \frac{\sec 2x}{2} + C \\ &= \frac{(\sec 2x)^3}{6} - \frac{\sec 2x}{2} + C\end{aligned}$$

Question 16:

$\tan^4 x$

Answer

$$\begin{aligned} & \tan^4 x \\ &= \tan^2 x \cdot \tan^2 x \\ &= (\sec^2 x - 1) \tan^2 x \\ &= \sec^2 x \tan^2 x - \tan^2 x \\ &= \sec^2 x \tan^2 x - (\sec^2 x - 1) \\ &= \sec^2 x \tan^2 x - \sec^2 x + 1 \end{aligned}$$

$$\begin{aligned} \therefore \int \tan^4 x \, dx &= \int \sec^2 x \tan^2 x \, dx - \int \sec^2 x \, dx + \int 1 \cdot dx \\ &= \int \sec^2 x \tan^2 x \, dx - \tan x + x + C \quad \dots(1) \end{aligned}$$

Consider $\int \sec^2 x \tan^2 x \, dx$

Let $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$$\Rightarrow \int \sec^2 x \tan^2 x \, dx = \int t^2 \, dt = \frac{t^3}{3} = \frac{\tan^3 x}{3}$$

From equation (1), we obtain

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C$$

Question 17:

$$\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$$

Answer

$$\begin{aligned} \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} &= \frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x} \\ &= \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x} \\ &= \tan x \sec x + \cot x \operatorname{cosec} x \end{aligned}$$

$$\begin{aligned} \therefore \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} \, dx &= \int (\tan x \sec x + \cot x \operatorname{cosec} x) \, dx \\ &= \sec x - \operatorname{cosec} x + C \end{aligned}$$

Question 18:

$$\frac{\cos 2x + 2 \sin^2 x}{\cos^2 x}$$

Answer

$$\begin{aligned} & \frac{\cos 2x + 2 \sin^2 x}{\cos^2 x} \\ &= \frac{\cos 2x + (1 - \cos 2x)}{\cos^2 x} \quad [\cos 2x = 1 - 2 \sin^2 x] \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x \end{aligned}$$

$$\therefore \int \frac{\cos 2x + 2 \sin^2 x}{\cos^2 x} dx = \int \sec^2 x dx = \tan x + C$$

Question 19:

$$\frac{1}{\sin x \cos^3 x}$$

Answer

$$\begin{aligned} \frac{1}{\sin x \cos^3 x} &= \frac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x} \\ &= \frac{\sin x}{\cos^3 x} + \frac{1}{\sin x \cos x} \\ &= \tan x \sec^2 x + \frac{1 \cos^2 x}{\sin x \cos x \cos^2 x} \\ &= \tan x \sec^2 x + \frac{\sec^2 x}{\tan x} \end{aligned}$$

$$\therefore \int \frac{1}{\sin x \cos^3 x} dx = \int \tan x \sec^2 x dx + \int \frac{\sec^2 x}{\tan x} dx$$

$$\text{Let } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\begin{aligned} \Rightarrow \int \frac{1}{\sin x \cos^3 x} dx &= \int t dt + \int \frac{1}{t} dt \\ &= \frac{t^2}{2} + \log|t| + C \\ &= \frac{1}{2} \tan^2 x + \log|\tan x| + C \end{aligned}$$

Question 20:

$$\frac{\cos 2x}{(\cos x + \sin x)^2}$$

Answer

$$\frac{\cos 2x}{(\cos x + \sin x)^2} = \frac{\cos 2x}{\cos^2 x + \sin^2 x + 2 \sin x \cos x} = \frac{\cos 2x}{1 + \sin 2x}$$

$$\therefore \int \frac{\cos 2x}{(\cos x + \sin x)^2} dx = \int \frac{\cos 2x}{(1 + \sin 2x)} dx$$

$$\text{Let } 1 + \sin 2x = t$$

$$\Rightarrow 2 \cos 2x dx = dt$$

$$\begin{aligned} \therefore \int \frac{\cos 2x}{(\cos x + \sin x)^2} dx &= \frac{1}{2} \int \frac{1}{t} dt \\ &= \frac{1}{2} \log |t| + C \\ &= \frac{1}{2} \log |1 + \sin 2x| + C \\ &= \frac{1}{2} \log |(\sin x + \cos x)^2| + C \\ &= \log |\sin x + \cos x| + C \end{aligned}$$

Question 21:

$$\sin^{-1}(\cos x)$$

Answer

$$\sin^{-1}(\cos x)$$

$$\text{Let } \cos x = t$$

$$\text{Then, } \sin x = \sqrt{1 - t^2}$$

$$\Rightarrow (-\sin x) dx = dt$$

$$dx = \frac{-dt}{\sin x}$$

$$dx = \frac{-dt}{\sqrt{1-t^2}}$$

$$\begin{aligned} \therefore \int \sin^{-1}(\cos x) dx &= \int \sin^{-1} t \left(\frac{-dt}{\sqrt{1-t^2}} \right) \\ &= - \int \frac{\sin^{-1} t}{\sqrt{1-t^2}} dt \end{aligned}$$

Let $\sin^{-1} t = u$

$$\Rightarrow \frac{1}{\sqrt{1-t^2}} dt = du$$

$$\begin{aligned} \therefore \int \sin^{-1}(\cos x) dx &= \int -u du \\ &= -\frac{u^2}{2} + C \\ &= \frac{-(\sin^{-1} t)^2}{2} + C \\ &= \frac{-[\sin^{-1}(\cos x)]^2}{2} + C \end{aligned}$$

It is known that,

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\therefore \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x) = \left(\frac{\pi}{2} - x \right)$$

Substituting in equation (1), we obtain

$$\begin{aligned} \dots(1) \quad \int \sin^{-1}(\cos x) dx &= \frac{-\left[\frac{\pi}{2} - x\right]^2}{2} + C \\ &= -\frac{1}{2} \left(\frac{\pi^2}{2} + x^2 - \pi x \right) + C \\ &= -\frac{\pi^2}{8} - \frac{x^2}{2} + \frac{1}{2} \pi x + C \\ &= \frac{\pi x}{2} - \frac{x^2}{2} + \left(C - \frac{\pi^2}{8} \right) \\ &= \frac{\pi x}{2} - \frac{x^2}{2} + C_1 \end{aligned}$$

Question 22:

$$\frac{1}{\cos(x-a)\cos(x-b)}$$

Answer

$$\begin{aligned} \frac{1}{\cos(x-a)\cos(x-b)} &= \frac{1}{\sin(a-b)} \left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)} \right] \\ &= \frac{1}{\sin(a-b)} \left[\frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)} \right] \\ &= \frac{1}{\sin(a-b)} \frac{[\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)]}{\cos(x-a)\cos(x-b)} \\ &= \frac{1}{\sin(a-b)} [\tan(x-b) - \tan(x-a)] \end{aligned}$$

$$\begin{aligned} \Rightarrow \int \frac{1}{\cos(x-a)\cos(x-b)} dx &= \frac{1}{\sin(a-b)} \int [\tan(x-b) - \tan(x-a)] dx \\ &= \frac{1}{\sin(a-b)} [-\log|\cos(x-b)| + \log|\cos(x-a)|] \\ &= \frac{1}{\sin(a-b)} \left[\log \left| \frac{\cos(x-a)}{\cos(x-b)} \right| \right] + C \end{aligned}$$

Question 23:

$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$ is equal to

- A. $\tan x + \cot x + C$
- B. $\tan x + \operatorname{cosec} x + C$
- C. $-\tan x + \cot x + C$
- D. $\tan x + \sec x + C$

Answer

$$\begin{aligned} \int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx &= \int \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} - \frac{\cos^2 x}{\sin^2 x \cos^2 x} \right) dx \\ &= \int (\sec^2 x - \operatorname{cosec}^2 x) dx \\ &= \tan x + \cot x + C \end{aligned}$$

Hence, the correct Answer is A.

Question 24:

$$\int \frac{e^x(1+x)}{\cos^2(e^x)} dx \quad \text{equals}$$

- A. $-\cot(e^{x^2}) + C$
- B. $\tan(xe^x) + C$
- C. $\tan(e^x) + C$
- D. $\cot(e^x) + C$

Answer

$$\int \frac{e^x(1+x)}{\cos^2(e^x)} dx$$

Let $ex^x = t$

$$\Rightarrow (e^x \cdot x + e^x \cdot 1) dx = dt$$

$$e^x(x+1) dx = dt$$

$$\begin{aligned} \therefore \int \frac{e^x(1+x)}{\cos^2(e^x)} dx &= \int \frac{dt}{\cos^2 t} \\ &= \int \sec^2 t dt \\ &= \tan t + C \\ &= \tan(e^x \cdot x) + C \end{aligned}$$

Hence, the correct Answer is B.

Exercise 7.4

Question 1:

$$\frac{3x^2}{x^6 + 1}$$

Answer

Let $x^3 = t$

$\therefore 3x^2 dx = dt$

$$\begin{aligned}\Rightarrow \int \frac{3x^2}{x^6 + 1} dx &= \int \frac{dt}{t^2 + 1} \\ &= \tan^{-1} t + C \\ &= \tan^{-1}(x^3) + C\end{aligned}$$

Question 2:

$$\frac{1}{\sqrt{1+4x^2}}$$

Answer

Let $2x = t$

$\therefore 2dx = dt$

$$\begin{aligned}\Rightarrow \int \frac{1}{\sqrt{1+4x^2}} dx &= \frac{1}{2} \int \frac{dt}{\sqrt{1+t^2}} \\ &= \frac{1}{2} \left[\log \left| t + \sqrt{t^2 + 1} \right| \right] + C && \left[\int \frac{1}{\sqrt{x^2 + a^2}} dt = \log \left| x + \sqrt{x^2 + a^2} \right| \right] \\ &= \frac{1}{2} \log \left| 2x + \sqrt{4x^2 + 1} \right| + C\end{aligned}$$

Question 3:

$$\frac{1}{\sqrt{(2-x)^2 + 1}}$$

Answer

Let $2 - x = t$

$$\Rightarrow -dx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}} dx &= -\int \frac{1}{\sqrt{t^2+1}} dt \\ &= -\log |t + \sqrt{t^2+1}| + C \quad \left[\int \frac{1}{\sqrt{x^2+a^2}} dt = \log |x + \sqrt{x^2+a^2}| \right] \\ &= -\log |2-x + \sqrt{(2-x)^2+1}| + C \\ &= \log \left| \frac{1}{(2-x) + \sqrt{x^2-4x+5}} \right| + C\end{aligned}$$

Question 4:

$$\frac{1}{\sqrt{9-25x^2}}$$

Answer

Let $5x = t$

$$\therefore 5dx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{1}{\sqrt{9-25x^2}} dx &= \frac{1}{5} \int \frac{1}{9-t^2} dt \\ &= \frac{1}{5} \int \frac{1}{\sqrt{3^2-t^2}} dt \\ &= \frac{1}{5} \sin^{-1}\left(\frac{t}{3}\right) + C \\ &= \frac{1}{5} \sin^{-1}\left(\frac{5x}{3}\right) + C\end{aligned}$$

Question 5:

$$\frac{3x}{1+2x^4}$$

Answer

$$\text{Let } \sqrt{2}x^2 = t$$

$$\therefore 2\sqrt{2}x dx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{3x}{1+2x^4} dx &= \frac{3}{2\sqrt{2}} \int \frac{dt}{1+t^2} \\ &= \frac{3}{2\sqrt{2}} [\tan^{-1} t] + C \\ &= \frac{3}{2\sqrt{2}} \tan^{-1}(\sqrt{2}x^2) + C\end{aligned}$$

Question 6:

$$\frac{x^2}{1-x^6}$$

Answer

$$\text{Let } x^3 = t$$

$$\therefore 3x^2 dx = dt$$

$$\begin{aligned} \Rightarrow \int \frac{x^2}{1-x^6} dx &= \frac{1}{3} \int \frac{dt}{1-t^2} \\ &= \frac{1}{3} \left[\frac{1}{2} \log \left| \frac{1+t}{1-t} \right| \right] + C \\ &= \frac{1}{6} \log \left| \frac{1+x^3}{1-x^3} \right| + C \end{aligned}$$

Question 7:

$$\frac{x-1}{\sqrt{x^2-1}}$$

Answer

$$\int \frac{x-1}{\sqrt{x^2-1}} dx = \int \frac{x}{\sqrt{x^2-1}} dx - \int \frac{1}{\sqrt{x^2-1}} dx \quad \dots(1)$$

For $\int \frac{x}{\sqrt{x^2-1}} dx$, let $x^2-1=t \Rightarrow 2x dx = dt$

$$\begin{aligned} \therefore \int \frac{x}{\sqrt{x^2-1}} dx &= \frac{1}{2} \int \frac{dt}{\sqrt{t}} \\ &= \frac{1}{2} \int t^{-\frac{1}{2}} dt \\ &= \frac{1}{2} \left[2t^{\frac{1}{2}} \right] \\ &= \sqrt{t} \\ &= \sqrt{x^2-1} \end{aligned}$$

From (1), we obtain

$$\begin{aligned} \int \frac{x-1}{\sqrt{x^2-1}} dx &= \int \frac{x}{\sqrt{x^2-1}} dx - \int \frac{1}{\sqrt{x^2-1}} dx && \left[\int \frac{1}{\sqrt{x^2-a^2}} dt = \log \left| x + \sqrt{x^2-a^2} \right| \right] \\ &= \sqrt{x^2-1} - \log \left| x + \sqrt{x^2-1} \right| + C \end{aligned}$$

Question 8:

$$\frac{x^2}{\sqrt{x^6 + a^6}}$$

Answer

Let $x^3 = t$

$$\Rightarrow 3x^2 dx = dt$$

$$\begin{aligned}\therefore \int \frac{x^2}{\sqrt{x^6 + a^6}} dx &= \frac{1}{3} \int \frac{dt}{\sqrt{t^2 + (a^3)^2}} \\ &= \frac{1}{3} \log |t + \sqrt{t^2 + a^6}| + C \\ &= \frac{1}{3} \log |x^3 + \sqrt{x^6 + a^6}| + C\end{aligned}$$

Question 9:

$$\frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}$$

Answer

Let $\tan x = t$

$$\therefore \sec^2 x dx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx &= \int \frac{dt}{\sqrt{t^2 + 2^2}} \\ &= \log |t + \sqrt{t^2 + 4}| + C \\ &= \log |\tan x + \sqrt{\tan^2 x + 4}| + C\end{aligned}$$

Question 10:

$$\frac{1}{\sqrt{x^2 + 2x + 2}}$$

Answer

$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{(x+1)^2 + (1)^2}} dx$$

Let $x+1 = t$

$\therefore dx = dt$

$$\begin{aligned} \Rightarrow \int \frac{1}{\sqrt{x^2 + 2x + 2}} dx &= \int \frac{1}{\sqrt{t^2 + 1}} dt \\ &= \log \left| t + \sqrt{t^2 + 1} \right| + C \\ &= \log \left| (x+1) + \sqrt{(x+1)^2 + 1} \right| + C \\ &= \log \left| (x+1) + \sqrt{x^2 + 2x + 2} \right| + C \end{aligned}$$

Question 11:

$$\frac{1}{\sqrt{9x^2 + 6x + 5}}$$

Answer

$$\int \frac{1}{9x^2 + 6x + 5} dx = \int \frac{1}{(3x+1)^2 + (2)^2} dx$$

$$\text{Let } (3x+1) = t$$

$$\therefore 3dx = dt$$

$$\begin{aligned} \Rightarrow \int \frac{1}{(3x+1)^2 + (2)^2} dx &= \frac{1}{3} \int \frac{1}{t^2 + 2^2} dt \\ &= \frac{1}{3} \left[\frac{1}{2} \tan^{-1} \left(\frac{t}{2} \right) \right] + C \\ &= \frac{1}{6} \tan^{-1} \left(\frac{3x+1}{2} \right) + C \end{aligned}$$

Question 12:

$$\frac{1}{\sqrt{7-6x-x^2}}$$

Answer

$$7-6x-x^2 \text{ can be written as } 7-(x^2+6x+9-9).$$

Therefore,

$$7-(x^2+6x+9-9)$$

$$= 16-(x^2+6x+9)$$

$$= 16-(x+3)^2$$

$$= (4)^2 - (x+3)^2$$

$$\therefore \int \frac{1}{\sqrt{7-6x-x^2}} dx = \int \frac{1}{\sqrt{(4)^2 - (x+3)^2}} dx$$

$$\text{Let } x+3 = t$$

$$\Rightarrow dx = dt$$

$$\begin{aligned} \Rightarrow \int \frac{1}{\sqrt{(4)^2 - (x+3)^2}} dx &= \int \frac{1}{\sqrt{(4)^2 - (t)^2}} dt \\ &= \sin^{-1} \left(\frac{t}{4} \right) + C \\ &= \sin^{-1} \left(\frac{x+3}{4} \right) + C \end{aligned}$$

Question 13:

$$\frac{1}{\sqrt{(x-1)(x-2)}}$$

Answer

$(x-1)(x-2)$ can be written as $x^2 - 3x + 2$.

Therefore,

$$\begin{aligned} & x^2 - 3x + 2 \\ &= x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2 \\ &= \left(x - \frac{3}{2}\right)^2 - \frac{1}{4} \\ &= \left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2 \end{aligned}$$

$$\therefore \int \frac{1}{\sqrt{(x-1)(x-2)}} dx = \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$$

$$\text{Let } x - \frac{3}{2} = t$$

$$\therefore dx = dt$$

$$\begin{aligned} \Rightarrow \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx &= \int \frac{1}{\sqrt{t^2 - \left(\frac{1}{2}\right)^2}} dt \\ &= \log \left| t + \sqrt{t^2 - \left(\frac{1}{2}\right)^2} \right| + C \\ &= \log \left| \left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 2} \right| + C \end{aligned}$$

Question 14:

$$\frac{1}{\sqrt{8+3x-x^2}}$$

Answer

$$8+3x-x^2 \text{ can be written as } 8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right).$$

Therefore,

$$8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

$$= \frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

$$\Rightarrow \int \frac{1}{\sqrt{8+3x-x^2}} dx = \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}} dx$$

$$\text{Let } x-\frac{3}{2} = t$$

$$\therefore dx = dt$$

$$\Rightarrow \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}} dx = \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-t^2}} dt$$

$$= \sin^{-1} \left(\frac{t}{\frac{\sqrt{41}}{2}} \right) + C$$

$$= \sin^{-1} \left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}} \right) + C$$

$$= \sin^{-1} \left(\frac{2x-3}{\sqrt{41}} \right) + C$$

Question 15:

$$\frac{1}{\sqrt{(x-a)(x-b)}}$$

Answer

$(x-a)(x-b)$ can be written as $x^2 - (a+b)x + ab$.

Therefore,

$$x^2 - (a+b)x + ab$$

$$= x^2 - (a+b)x + \frac{(a+b)^2}{4} - \frac{(a+b)^2}{4} + ab$$

$$= \left[x - \left(\frac{a+b}{2} \right) \right]^2 - \frac{(a-b)^2}{4}$$

$$\Rightarrow \int \frac{1}{\sqrt{(x-a)(x-b)}} dx = \int \frac{1}{\sqrt{\left\{ x - \left(\frac{a+b}{2} \right) \right\}^2 - \left(\frac{a-b}{2} \right)^2}} dx$$

$$\text{Let } x - \left(\frac{a+b}{2} \right) = t$$

$$\therefore dx = dt$$

$$\Rightarrow \int \frac{1}{\sqrt{\left\{ x - \left(\frac{a+b}{2} \right) \right\}^2 - \left(\frac{a-b}{2} \right)^2}} dx = \int \frac{1}{\sqrt{t^2 - \left(\frac{a-b}{2} \right)^2}} dt$$

$$= \log \left| t + \sqrt{t^2 - \left(\frac{a-b}{2} \right)^2} \right| + C$$

$$= \log \left| \left\{ x - \left(\frac{a+b}{2} \right) \right\} + \sqrt{(x-a)(x-b)} \right| + C$$

Question 16:

$$\frac{4x+1}{\sqrt{2x^2+x-3}}$$

Answer

$$\text{Let } 4x+1 = A \frac{d}{dx}(2x^2+x-3) + B$$

$$\Rightarrow 4x+1 = A(4x+1) + B$$

$$\Rightarrow 4x+1 = 4Ax + A + B$$

Equating the coefficients of x and constant term on both sides, we obtain $4A = 4 \Rightarrow A = 1$

$$A + B = 1 \Rightarrow B = 0$$

$$\text{Let } 2x^2 + x - 3 = t$$

$$\therefore (4x + 1) dx = dt$$

$$\begin{aligned} \Rightarrow \int \frac{4x+1}{\sqrt{2x^2+x-3}} dx &= \int \frac{1}{\sqrt{t}} dt \\ &= 2\sqrt{t} + C \\ &= 2\sqrt{2x^2+x-3} + C \end{aligned}$$

Question 17:

$$\frac{x+2}{\sqrt{x^2-1}}$$

Answer

$$\text{Let } x+2 = A \frac{d}{dx}(x^2-1) + B \quad \dots(1)$$

$$\Rightarrow x+2 = A(2x) + B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$B = 2$$

From (1), we obtain

$$(x+2) = \frac{1}{2}(2x) + 2$$

$$\begin{aligned} \text{Then, } \int \frac{x+2}{\sqrt{x^2-1}} dx &= \int \frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}} dx \\ &= \frac{1}{2} \int \frac{2x}{\sqrt{x^2-1}} dx + \int \frac{2}{\sqrt{x^2-1}} dx \quad \dots(2) \end{aligned}$$

$$\text{In } \frac{1}{2} \int \frac{2x}{\sqrt{x^2-1}} dx, \text{ let } x^2-1=t \Rightarrow 2x dx = dt$$

$$\begin{aligned} \frac{1}{2} \int \frac{2x}{\sqrt{x^2-1}} dx &= \frac{1}{2} \int \frac{dt}{\sqrt{t}} \\ &= \frac{1}{2} [2\sqrt{t}] \\ &= \sqrt{t} \\ &= \sqrt{x^2-1} \end{aligned}$$

$$\text{Then, } \int \frac{2}{\sqrt{x^2-1}} dx = 2 \int \frac{1}{\sqrt{x^2-1}} dx = 2 \log |x + \sqrt{x^2-1}|$$

From equation (2), we obtain

$$\int \frac{x+2}{\sqrt{x^2-1}} dx = \sqrt{x^2-1} + 2 \log |x + \sqrt{x^2-1}| + C$$

Question 18:

$$\frac{5x-2}{1+2x+3x^2}$$

Answer

$$\text{Let } 5x-2 = A \frac{d}{dx}(1+2x+3x^2) + B$$

$$\Rightarrow 5x-2 = A(2+6x) + B$$

Equating the coefficient of x and constant term on both sides, we obtain

$$5 = 6A \Rightarrow A = \frac{5}{6}$$

$$2A + B = -2 \Rightarrow B = -\frac{11}{3}$$

$$\therefore 5x - 2 = \frac{5}{6}(2 + 6x) + \left(-\frac{11}{3}\right)$$

$$\begin{aligned} \Rightarrow \int \frac{5x-2}{1+2x+3x^2} dx &= \int \frac{\frac{5}{6}(2+6x) - \frac{11}{3}}{1+2x+3x^2} dx \\ &= \frac{5}{6} \int \frac{2+6x}{1+2x+3x^2} dx - \frac{11}{3} \int \frac{1}{1+2x+3x^2} dx \end{aligned}$$

$$\text{Let } I_1 = \int \frac{2+6x}{1+2x+3x^2} dx \text{ and } I_2 = \int \frac{1}{1+2x+3x^2} dx$$

$$\therefore \int \frac{5x-2}{1+2x+3x^2} dx = \frac{5}{6} I_1 - \frac{11}{3} I_2 \quad \dots(1)$$

$$I_1 = \int \frac{2+6x}{1+2x+3x^2} dx$$

$$\text{Let } 1+2x+3x^2 = t$$

$$\Rightarrow (2+6x) dx = dt$$

$$\therefore I_1 = \int \frac{dt}{t}$$

$$I_1 = \log|t|$$

$$I_1 = \log|1+2x+3x^2| \quad \dots(2)$$

$$I_2 = \int \frac{1}{1+2x+3x^2} dx$$

$1+2x+3x^2$ can be written as $1+3\left(x^2+\frac{2}{3}x\right)$.

Therefore,

$$\begin{aligned} &1+3\left(x^2+\frac{2}{3}x\right) \\ &=1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right) \\ &=1+3\left(x+\frac{1}{3}\right)^2-\frac{1}{3} \\ &=\frac{2}{3}+3\left(x+\frac{1}{3}\right)^2 \\ &=3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right] \\ &=3\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right] \end{aligned}$$

$$\begin{aligned} I_2 &= \frac{1}{3} \int \frac{1}{\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]} dx \\ &= \frac{1}{3} \left[\frac{1}{\frac{\sqrt{2}}{3}} \tan^{-1} \left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}} \right) \right] \\ &= \frac{1}{3} \left[\frac{3}{\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) \right] \\ &= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) \quad \dots(3) \end{aligned}$$

Substituting equations (2) and (3) in equation (1), we obtain

$$\int \frac{5x-2}{1+2x+3x^2} dx = \frac{5}{6} \left[\log |1+2x+3x^2| \right] - \frac{11}{3} \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) \right] + C$$
$$= \frac{5}{6} \log |1+2x+3x^2| - \frac{11}{3\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) + C$$

Question 19:

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}}$$

Answer

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}} = \frac{6x+7}{\sqrt{x^2-9x+20}}$$

$$\text{Let } 6x+7 = A \frac{d}{dx} (x^2-9x+20) + B$$

$$\Rightarrow 6x+7 = A(2x-9) + B$$

Equating the coefficients of x and constant term, we obtain

$$2A = 6 \Rightarrow A = 3$$

$$-9A + B = 7 \Rightarrow B = 34$$

$$\therefore 6x + 7 = 3(2x - 9) + 34$$

$$\begin{aligned}\int \frac{6x+7}{\sqrt{x^2-9x+20}} &= \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}} dx \\ &= 3 \int \frac{2x-9}{\sqrt{x^2-9x+20}} dx + 34 \int \frac{1}{\sqrt{x^2-9x+20}} dx\end{aligned}$$

$$\text{Let } I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}} dx \text{ and } I_2 = \int \frac{1}{\sqrt{x^2-9x+20}} dx$$

$$\therefore \int \frac{6x+7}{\sqrt{x^2-9x+20}} = 3I_1 + 34I_2 \quad \dots(1)$$

Then,

$$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}} dx$$

$$\text{Let } x^2 - 9x + 20 = t$$

$$\Rightarrow (2x-9) dx = dt$$

$$\Rightarrow I_1 = \frac{dt}{\sqrt{t}}$$

$$I_1 = 2\sqrt{t}$$

$$I_1 = 2\sqrt{x^2 - 9x + 20} \quad \dots(2)$$

$$\text{and } I_2 = \int \frac{1}{\sqrt{x^2-9x+20}} dx$$

$x^2 - 9x + 20$ can be written as $x^2 - 9x + 20 + \frac{81}{4} - \frac{81}{4}$.

Therefore,

$$x^2 - 9x + 20 + \frac{81}{4} - \frac{81}{4}$$

$$= \left(x - \frac{9}{2}\right)^2 - \frac{1}{4}$$

$$= \left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$\Rightarrow I_2 = \int \frac{1}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$$

$$I_2 = \log \left| \left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20} \right| \quad \dots(3)$$

Substituting equations (2) and (3) in (1), we obtain

$$\begin{aligned} \int \frac{6x+7}{\sqrt{x^2-9x+20}} dx &= 3 \left[2\sqrt{x^2-9x+20} \right] + 34 \log \left[\left(x - \frac{9}{2}\right) + \sqrt{x^2-9x+20} \right] + C \\ &= 6\sqrt{x^2-9x+20} + 34 \log \left[\left(x - \frac{9}{2}\right) + \sqrt{x^2-9x+20} \right] + C \end{aligned}$$

Question 20:

$$\frac{x+2}{\sqrt{4x-x^2}}$$

Answer

$$\text{Let } x+2 = A \frac{d}{dx}(4x-x^2) + B$$

$$\Rightarrow x+2 = A(4-2x) + B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$-2A = 1 \Rightarrow A = -\frac{1}{2}$$

$$4A + B = 2 \Rightarrow B = 4$$

$$\Rightarrow (x+2) = -\frac{1}{2}(4-2x) + 4$$

$$\begin{aligned} \therefore \int \frac{x+2}{\sqrt{4x-x^2}} dx &= \int \frac{-\frac{1}{2}(4-2x) + 4}{\sqrt{4x-x^2}} dx \\ &= -\frac{1}{2} \int \frac{4-2x}{\sqrt{4x-x^2}} dx + 4 \int \frac{1}{\sqrt{4x-x^2}} dx \end{aligned}$$

$$\text{Let } I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}} dx \text{ and } I_2 = \int \frac{1}{\sqrt{4x-x^2}} dx$$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}} dx = -\frac{1}{2} I_1 + 4 I_2 \quad \dots(1)$$

$$\text{Then, } I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}} dx$$

$$\text{Let } 4x-x^2 = t$$

$$\Rightarrow (4-2x) dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{4x-x^2} \quad \dots(2)$$

$$I_2 = \int \frac{1}{\sqrt{4x-x^2}} dx$$

$$\begin{aligned} \Rightarrow 4x-x^2 &= -(-4x+x^2) \\ &= (-4x+x^2+4-4) \\ &= 4-(x-2)^2 \\ &= (2)^2-(x-2)^2 \end{aligned}$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(2)^2-(x-2)^2}} dx = \sin^{-1} \left(\frac{x-2}{2} \right) \quad \dots(3)$$

Using equations (2) and (3) in (1), we obtain

$$\begin{aligned}\int \frac{x+2}{\sqrt{4x-x^2}} dx &= -\frac{1}{2} \left(2\sqrt{4x-x^2} \right) + 4 \sin^{-1} \left(\frac{x-2}{2} \right) + C \\ &= -\sqrt{4x-x^2} + 4 \sin^{-1} \left(\frac{x-2}{2} \right) + C\end{aligned}$$

Question 21:

$$\frac{x+2}{\sqrt{x^2+2x+3}}$$

Answer

$$\begin{aligned}\int \frac{(x+2)}{\sqrt{x^2+2x+3}} dx &= \frac{1}{2} \int \frac{2(x+2)}{\sqrt{x^2+2x+3}} dx \\ &= \frac{1}{2} \int \frac{2x+4}{\sqrt{x^2+2x+3}} dx \\ &= \frac{1}{2} \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \frac{1}{2} \int \frac{2}{\sqrt{x^2+2x+3}} dx \\ &= \frac{1}{2} \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \int \frac{1}{\sqrt{x^2+2x+3}} dx\end{aligned}$$

Let $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+2x+3}} dx$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}} dx = \frac{1}{2} I_1 + I_2 \quad \dots(1)$$

Then, $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$

Let $x^2 + 2x + 3 = t$

$\Rightarrow (2x + 2) dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2 + 2x + 3} \quad \dots(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2 + 2x + 3}} dx$$

$$\Rightarrow x^2 + 2x + 3 = x^2 + 2x + 1 + 2 = (x+1)^2 + (\sqrt{2})^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2 + (\sqrt{2})^2}} dx = \log \left| (x+1) + \sqrt{x^2 + 2x + 3} \right| \quad \dots(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{x^2 + 2x + 3}} dx = \frac{1}{2} \left[2\sqrt{x^2 + 2x + 3} \right] + \log \left| (x+1) + \sqrt{x^2 + 2x + 3} \right| + C$$

$$= \sqrt{x^2 + 2x + 3} + \log \left| (x+1) + \sqrt{x^2 + 2x + 3} \right| + C$$

Question 22:

$$\frac{x+3}{x^2 - 2x - 5}$$

Answer

$$\text{Let } (x+3) = A \frac{d}{dx} (x^2 - 2x - 5) + B$$

$$(x+3) = A(2x-2) + B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$-2A + B = 3 \Rightarrow B = 4$$

$$\therefore (x+3) = \frac{1}{2}(2x-2) + 4$$

$$\Rightarrow \int \frac{x+3}{x^2 - 2x - 5} dx = \int \frac{\frac{1}{2}(2x-2) + 4}{x^2 - 2x - 5} dx$$

$$= \frac{1}{2} \int \frac{2x-2}{x^2 - 2x - 5} dx + 4 \int \frac{1}{x^2 - 2x - 5} dx$$

$$\text{Let } I_1 = \int \frac{2x-2}{x^2-2x-5} dx \text{ and } I_2 = \int \frac{1}{x^2-2x-5} dx$$

$$\therefore \int \frac{x+3}{(x^2-2x-5)} dx = \frac{1}{2} I_1 + 4I_2 \quad \dots(1)$$

$$\text{Then, } I_1 = \int \frac{2x-2}{x^2-2x-5} dx$$

$$\text{Let } x^2 - 2x - 5 = t$$

$$\Rightarrow (2x-2) dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{t} = \log|t| = \log|x^2 - 2x - 5| \quad \dots(2)$$

$$I_2 = \int \frac{1}{x^2-2x-5} dx$$

$$= \int \frac{1}{(x^2-2x+1)-6} dx$$

$$= \int \frac{1}{(x-1)^2 + (\sqrt{6})^2} dx$$

$$= \frac{1}{2\sqrt{6}} \log \left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}} \right) \quad \dots(3)$$

Substituting (2) and (3) in (1), we obtain

$$\int \frac{x+3}{x^2-2x-5} dx = \frac{1}{2} \log|x^2-2x-5| + \frac{4}{2\sqrt{6}} \log \left| \frac{x-1-\sqrt{6}}{x-1+\sqrt{6}} \right| + C$$

$$= \frac{1}{2} \log|x^2-2x-5| + \frac{2}{\sqrt{6}} \log \left| \frac{x-1-\sqrt{6}}{x-1+\sqrt{6}} \right| + C$$

Question 23:

$$\frac{5x+3}{\sqrt{x^2+4x+10}}$$

Answer

$$\text{Let } 5x+3 = A \frac{d}{dx}(x^2+4x+10) + B$$

$$\Rightarrow 5x+3 = A(2x+4) + B$$

Equating the coefficients of x and constant term, we obtain

$$2A = 5 \Rightarrow A = \frac{5}{2}$$

$$4A + B = 3 \Rightarrow B = -7$$

$$\therefore 5x+3 = \frac{5}{2}(2x+4) - 7$$

$$\begin{aligned} \Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}} dx &= \int \frac{\frac{5}{2}(2x+4) - 7}{\sqrt{x^2+4x+10}} dx \\ &= \frac{5}{2} \int \frac{2x+4}{\sqrt{x^2+4x+10}} dx - 7 \int \frac{1}{\sqrt{x^2+4x+10}} dx \end{aligned}$$

$$\text{Let } I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}} dx \text{ and } I_2 = \int \frac{1}{\sqrt{x^2+4x+10}} dx$$

$$\therefore \int \frac{5x+3}{\sqrt{x^2+4x+10}} dx = \frac{5}{2} I_1 - 7 I_2 \quad \dots(1)$$

$$\text{Then, } I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}} dx$$

$$\text{Let } x^2+4x+10 = t$$

$$\therefore (2x+4) dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{t} = 2\sqrt{t} = 2\sqrt{x^2+4x+10} \quad \dots(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+4x+10}} dx$$

$$= \int \frac{1}{\sqrt{(x^2+4x+4)+6}} dx$$

$$= \int \frac{1}{(x+2)^2 + (\sqrt{6})^2} dx$$

$$= \log \left| (x+2)\sqrt{x^2+4x+10} \right| \quad \dots(3)$$

Using equations (2) and (3) in (1), we obtain

$$\begin{aligned}\int \frac{5x+3}{\sqrt{x^2+4x+10}} dx &= \frac{5}{2} \left[2\sqrt{x^2+4x+10} \right] - 7 \log \left| (x+2) + \sqrt{x^2+4x+10} \right| + C \\ &= 5\sqrt{x^2+4x+10} - 7 \log \left| (x+2) + \sqrt{x^2+4x+10} \right| + C\end{aligned}$$

Question 24:

$$\int \frac{dx}{x^2+2x+2} \text{ equals}$$

- A. $x \tan^{-1}(x+1) + C$
- B. $\tan^{-1}(x+1) + C$
- C. $(x+1) \tan^{-1} x + C$
- D. $\tan^{-1} x + C$

Answer

$$\begin{aligned}\int \frac{dx}{x^2+2x+2} &= \int \frac{dx}{(x^2+2x+1)+1} \\ &= \int \frac{1}{(x+1)^2+(1)^2} dx \\ &= \left[\tan^{-1}(x+1) \right] + C\end{aligned}$$

Hence, the correct Answer is B.

Question 25:

$$\int \frac{dx}{\sqrt{9x-4x^2}} \text{ equals}$$

A. $\frac{1}{9} \sin^{-1} \left(\frac{9x-8}{8} \right) + C$

B. $\frac{1}{2} \sin^{-1} \left(\frac{8x-9}{9} \right) + C$

C. $\frac{1}{3} \sin^{-1} \left(\frac{9x-8}{8} \right) + C$

$$D. \frac{1}{2} \sin^{-1} \left(\frac{9x-8}{9} \right) + C$$

Answer

$$\begin{aligned} & \int \frac{dx}{\sqrt{9x-4x^2}} \\ &= \int \frac{1}{\sqrt{-4 \left(x^2 - \frac{9}{4}x \right)}} dx \\ &= \int \frac{1}{-4 \left(x^2 - \frac{9}{4}x + \frac{81}{64} - \frac{81}{64} \right)} dx \\ &= \int \frac{1}{\sqrt{-4 \left[\left(x - \frac{9}{8} \right)^2 - \left(\frac{9}{8} \right)^2 \right]}} dx \\ &= \frac{1}{2} \int \frac{1}{\sqrt{\left(\frac{9}{8} \right)^2 - \left(x - \frac{9}{8} \right)^2}} dx \\ &= \frac{1}{2} \left[\sin^{-1} \left(\frac{x - \frac{9}{8}}{\frac{9}{8}} \right) \right] + C \\ &= \frac{1}{2} \sin^{-1} \left(\frac{8x-9}{9} \right) + C \end{aligned}$$

$$\left(\int \frac{dy}{\sqrt{a^2 - y^2}} = \sin^{-1} \frac{y}{a} + C \right)$$

Hence, the correct Answer is B.

Exercise 7.5

Question 1:

$$\frac{x}{(x+1)(x+2)}$$

Answer

$$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

Let

$$\Rightarrow x = A(x+2) + B(x+1)$$

Equating the coefficients of x and constant term, we obtain

$$A + B = 1$$

$$2A + B = 0$$

On solving, we obtain

$$A = -1 \text{ and } B = 2$$

$$\therefore \frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$$

$$\begin{aligned} \Rightarrow \int \frac{x}{(x+1)(x+2)} dx &= \int \frac{-1}{x+1} + \frac{2}{x+2} dx \\ &= -\log|x+1| + 2\log|x+2| + C \\ &= \log(x+2)^2 - \log|x+1| + C \\ &= \log \frac{(x+2)^2}{(x+1)} + C \end{aligned}$$

Question 2:

$$\frac{1}{x^2 - 9}$$

Answer

$$\frac{1}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3}$$

Let

$$1 = A(x-3) + B(x+3)$$

Equating the coefficients of x and constant term, we obtain

$$A + B = 0$$

$$-3A + 3B = 1$$

On solving, we obtain

$$A = -\frac{1}{6} \text{ and } B = \frac{1}{6}$$

$$\therefore \frac{1}{(x+3)(x-3)} = \frac{-1}{6(x+3)} + \frac{1}{6(x-3)}$$

$$\begin{aligned} \Rightarrow \int \frac{1}{(x^2 - 9)} dx &= \int \left(\frac{-1}{6(x+3)} + \frac{1}{6(x-3)} \right) dx \\ &= -\frac{1}{6} \log|x+3| + \frac{1}{6} \log|x-3| + C \\ &= \frac{1}{6} \log \left| \frac{(x-3)}{(x+3)} \right| + C \end{aligned}$$

Question 3:

$$\frac{3x-1}{(x-1)(x-2)(x-3)}$$

Answer

$$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

Let

$$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad \dots(1)$$

Substituting $x = 1, 2,$ and 3 respectively in equation (1), we obtain

$A = 1, B = -5,$ and $C = 4$

$$\therefore \frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)}$$

$$\begin{aligned} \Rightarrow \int \frac{3x-1}{(x-1)(x-2)(x-3)} dx &= \int \left\{ \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)} \right\} dx \\ &= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C \end{aligned}$$

Question 4:

$$\frac{x}{(x-1)(x-2)(x-3)}$$

Answer

$$\frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

Let

$$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad \dots(1)$$

Substituting $x = 1, 2,$ and 3 respectively in equation (1), we obtain

$$A = \frac{1}{2}, B = -2, \text{ and } C = \frac{3}{2}$$

$$\therefore \frac{x}{(x-1)(x-2)(x-3)} = \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)}$$

$$\begin{aligned} \Rightarrow \int \frac{x}{(x-1)(x-2)(x-3)} dx &= \int \left[\frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)} \right] dx \\ &= \frac{1}{2} \log|x-1| - 2 \log|x-2| + \frac{3}{2} \log|x-3| + C \end{aligned}$$

Question 5:

$$\frac{2x}{x^2 + 3x + 2}$$

Answer

$$\frac{2x}{x^2 + 3x + 2} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

Let

$$2x = A(x+2) + B(x+1) \quad \dots(1)$$

Substituting $x = -1$ and -2 in equation (1), we obtain

$$A = -2 \text{ and } B = 4$$

$$\begin{aligned} \therefore \frac{2x}{(x+1)(x+2)} &= \frac{-2}{(x+1)} + \frac{4}{(x+2)} \\ \Rightarrow \int \frac{2x}{(x+1)(x+2)} dx &= \int \left[\frac{4}{(x+2)} - \frac{2}{(x+1)} \right] dx \\ &= 4 \log|x+2| - 2 \log|x+1| + C \end{aligned}$$

Question 6:

$$\frac{1-x^2}{x(1-2x)}$$

Answer

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(1 - x^2)$ by $x(1 - 2x)$, we obtain

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2} \left(\frac{2-x}{x(1-2x)} \right)$$

$$\frac{2-x}{x(1-2x)} = \frac{A}{x} + \frac{B}{(1-2x)}$$

Let

$$\Rightarrow (2-x) = A(1-2x) + Bx \quad \dots(1)$$

Substituting $x = 0$ and $\frac{1}{2}$ in equation (1), we obtain

$$A = 2 \text{ and } B = 3$$

$$\therefore \frac{2-x}{x(1-2x)} = \frac{2}{x} + \frac{3}{1-2x}$$

Substituting in equation (1), we obtain

$$\begin{aligned} \frac{1-x^2}{x(1-2x)} &= \frac{1}{2} + \frac{1}{2} \left\{ \frac{2}{x} + \frac{3}{1-2x} \right\} \\ \Rightarrow \int \frac{1-x^2}{x(1-2x)} dx &= \int \left\{ \frac{1}{2} + \frac{1}{2} \left(\frac{2}{x} + \frac{3}{1-2x} \right) \right\} dx \\ &= \frac{x}{2} + \log|x| + \frac{3}{2(-2)} \log|1-2x| + C \\ &= \frac{x}{2} + \log|x| - \frac{3}{4} \log|1-2x| + C \end{aligned}$$

Question 7:

$$\frac{x}{(x^2+1)(x-1)}$$

Answer

$$\frac{x}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$$

Let

$$x = (Ax+B)(x-1) + C(x^2+1)$$

$$x = Ax^2 - Ax + Bx - B + Cx^2 + C$$

Equating the coefficients of x^2 , x , and constant term, we obtain

$$A + C = 0$$

$$-A + B = 1$$

$$-B + C = 0$$

On solving these equations, we obtain

$$A = -\frac{1}{2}, B = \frac{1}{2}, \text{ and } C = \frac{1}{2}$$

From equation (1), we obtain

$$\begin{aligned} \therefore \frac{x}{(x^2+1)(x-1)} &= \frac{\left(-\frac{1}{2}x + \frac{1}{2}\right)}{x^2+1} + \frac{1}{x-1} \\ \Rightarrow \int \frac{x}{(x^2+1)(x-1)} &= -\frac{1}{2} \int \frac{x}{x^2+1} dx + \frac{1}{2} \int \frac{1}{x^2+1} dx + \frac{1}{2} \int \frac{1}{x-1} dx \\ &= -\frac{1}{4} \int \frac{2x}{x^2+1} dx + \frac{1}{2} \tan^{-1} x + \frac{1}{2} \log|x-1| + C \end{aligned}$$

Consider $\int \frac{2x}{x^2+1} dx$, let $(x^2+1) = t \Rightarrow 2x dx = dt$

$$\Rightarrow \int \frac{2x}{x^2+1} dx = \int \frac{dt}{t} = \log|t| = \log|x^2+1|$$

$$\begin{aligned} \therefore \int \frac{x}{(x^2+1)(x-1)} &= -\frac{1}{4} \log|x^2+1| + \frac{1}{2} \tan^{-1} x + \frac{1}{2} \log|x-1| + C \\ &= \frac{1}{2} \log|x-1| - \frac{1}{4} \log|x^2+1| + \frac{1}{2} \tan^{-1} x + C \end{aligned}$$

Question 8:

$$\frac{x}{(x-1)^2(x+2)}$$

Answer

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

Let

$$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

Substituting $x = 1$, we obtain

$$B = \frac{1}{3}$$

Equating the coefficients of x^2 and constant term, we obtain

$$A + C = 0$$

$$-2A + 2B + C = 0$$

On solving, we obtain

$$A = \frac{2}{9} \text{ and } C = \frac{-2}{9}$$

$$\begin{aligned} \therefore \frac{x}{(x-1)^2(x+2)} &= \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)} \\ \Rightarrow \int \frac{x}{(x-1)^2(x+2)} dx &= \frac{2}{9} \int \frac{1}{(x-1)} dx + \frac{1}{3} \int \frac{1}{(x-1)^2} dx - \frac{2}{9} \int \frac{1}{(x+2)} dx \\ &= \frac{2}{9} \log|x-1| + \frac{1}{3} \left(\frac{-1}{x-1} \right) - \frac{2}{9} \log|x+2| + C \\ &= \frac{2}{9} \log \left| \frac{x-1}{x+2} \right| - \frac{1}{3(x-1)} + C \end{aligned}$$

Question 9:

$$\frac{3x+5}{x^3-x^2-x+1}$$

Answer

$$\frac{3x+5}{x^3-x^2-x+1} = \frac{3x+5}{(x-1)^2(x+1)}$$

$$\text{Let } \frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

$$3x+5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

$$3x+5 = A(x^2-1) + B(x+1) + C(x^2+1-2x) \quad \dots(1)$$

Substituting $x = 1$ in equation (1), we obtain

$$B = 4$$

Equating the coefficients of x^2 and x , we obtain

$$A + C = 0$$

$$B - 2C = 3$$

$$\begin{aligned} \therefore \frac{3x+5}{(x-1)^2(x+1)} &= \frac{-1}{2(x-1)} + \frac{4}{(x-1)^2} + \frac{1}{2(x+1)} \\ \Rightarrow \int \frac{3x+5}{(x-1)^2(x+1)} dx &= -\frac{1}{2} \int \frac{1}{x-1} dx + 4 \int \frac{1}{(x-1)^2} dx + \frac{1}{2} \int \frac{1}{(x+1)} dx \\ &= -\frac{1}{2} \log|x-1| + 4 \left(\frac{-1}{x-1} \right) + \frac{1}{2} \log|x+1| + C \\ &= \frac{1}{2} \log \left| \frac{x+1}{x-1} \right| - \frac{4}{(x-1)} + C \end{aligned}$$

On solving, we obtain

$$A = -\frac{1}{2} \text{ and } C = \frac{1}{2}$$

Question 10:

$$\frac{2x-3}{(x^2-1)(2x+3)}$$

Answer

$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x+1)(x-1)(2x+3)}$$

$$\text{Let } \frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{2x+3}$$

$$\Rightarrow (2x-3) = A(x-1)(2x+3) + B(x+1)(2x+3) + C(x+1)(x-1)$$

$$\Rightarrow (2x-3) = A(2x^2+x-3) + B(2x^2+5x+3) + C(x^2-1)$$

$$\Rightarrow (2x-3) = (2A+2B+C)x^2 + (A+5B)x + (-3A+3B-C)$$

Equating the coefficients of x^2 and x , we obtain

$$B = -\frac{1}{10}, A = \frac{5}{2}, \text{ and } C = -\frac{24}{5}$$

$$\begin{aligned} \therefore \frac{2x-3}{(x+1)(x-1)(2x+3)} &= \frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)} \\ \Rightarrow \int \frac{2x-3}{(x^2-1)(2x+3)} dx &= \frac{5}{2} \int \frac{1}{(x+1)} dx - \frac{1}{10} \int \frac{1}{x-1} dx - \frac{24}{5} \int \frac{1}{(2x+3)} dx \\ &= \frac{5}{2} \log|x+1| - \frac{1}{10} \log|x-1| - \frac{24}{5 \times 2} \log|2x+3| \\ &= \frac{5}{2} \log|x+1| - \frac{1}{10} \log|x-1| - \frac{12}{5} \log|2x+3| + C \end{aligned}$$

Question 11:

$$\frac{5x}{(x+1)(x^2-4)}$$

Answer

$$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)}$$

$$\text{Let } \frac{5x}{(x+1)(x+2)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-2)}$$

$$5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2) \quad \dots(1)$$

Substituting $x = -1, -2,$ and 2 respectively in equation (1), we obtain

$$A = \frac{5}{3}, B = -\frac{5}{2}, \text{ and } C = \frac{5}{6}$$

$$\begin{aligned} \therefore \frac{5x}{(x+1)(x+2)(x-2)} &= \frac{5}{3(x+1)} - \frac{5}{2(x+2)} + \frac{5}{6(x-2)} \\ \Rightarrow \int \frac{5x}{(x+1)(x^2-4)} dx &= \frac{5}{3} \int \frac{1}{(x+1)} dx - \frac{5}{2} \int \frac{1}{(x+2)} dx + \frac{5}{6} \int \frac{1}{(x-2)} dx \\ &= \frac{5}{3} \log|x+1| - \frac{5}{2} \log|x+2| + \frac{5}{6} \log|x-2| + C \end{aligned}$$

Question 12:

$$\frac{x^3 + x + 1}{x^2 - 1}$$

Answer

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(x^3 + x + 1)$ by $x^2 - 1$, we obtain

$$\frac{x^3 + x + 1}{x^2 - 1} = x + \frac{2x + 1}{x^2 - 1}$$

$$\frac{2x + 1}{x^2 - 1} = \frac{A}{(x + 1)} + \frac{B}{(x - 1)}$$

Let

$$2x + 1 = A(x - 1) + B(x + 1) \quad \dots(1)$$

Substituting $x = 1$ and -1 in equation (1), we obtain

$$A = \frac{1}{2} \text{ and } B = \frac{3}{2}$$

$$\therefore \frac{x^3 + x + 1}{x^2 - 1} = x + \frac{1}{2(x + 1)} + \frac{3}{2(x - 1)}$$

$$\begin{aligned} \Rightarrow \int \frac{x^3 + x + 1}{x^2 - 1} dx &= \int x dx + \frac{1}{2} \int \frac{1}{(x + 1)} dx + \frac{3}{2} \int \frac{1}{(x - 1)} dx \\ &= \frac{x^2}{2} + \frac{1}{2} \log|x + 1| + \frac{3}{2} \log|x - 1| + C \end{aligned}$$

Question 13:

$$\frac{2}{(1 - x)(1 + x^2)}$$

Answer

$$\text{Let } \frac{2}{(1 - x)(1 + x^2)} = \frac{A}{(1 - x)} + \frac{Bx + C}{(1 + x^2)}$$

$$2 = A(1 + x^2) + (Bx + C)(1 - x)$$

$$2 = A + Ax^2 + Bx - Bx^2 + C - Cx$$

Equating the coefficient of x^2 , x , and constant term, we obtain

$$A - B = 0$$

$$-C = 0$$

$$A + C = 2$$

On solving these equations, we obtain

$$A = 1, B = 1, \text{ and } C = 1$$

$$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

$$\begin{aligned} \Rightarrow \int \frac{2}{(1-x)(1+x^2)} dx &= \int \frac{1}{1-x} dx + \int \frac{x}{1+x^2} dx + \int \frac{1}{1+x^2} dx \\ &= -\int \frac{1}{x-1} dx + \frac{1}{2} \int \frac{2x}{1+x^2} dx + \int \frac{1}{1+x^2} dx \\ &= -\log|x-1| + \frac{1}{2} \log|1+x^2| + \tan^{-1} x + C \end{aligned}$$

Question 14:

$$\frac{3x-1}{(x+2)^2}$$

Answer

$$\text{Let } \frac{3x-1}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$\Rightarrow 3x-1 = A(x+2) + B$$

Equating the coefficient of x and constant term, we obtain

$$A = 3$$

$$2A + B = -1 \Rightarrow B = -7$$

$$\therefore \frac{3x-1}{(x+2)^2} = \frac{3}{x+2} - \frac{7}{(x+2)^2}$$

$$\Rightarrow \int \frac{3x-1}{(x+2)^2} dx = 3 \int \frac{1}{x+2} dx - 7 \int \frac{x}{(x+2)^2} dx$$

$$= 3 \log|x+2| - 7 \left(\frac{-1}{x+2} \right) + C$$

$$= 3 \log|x+2| + \frac{7}{x+2} + C$$

Question 15:

$$\frac{1}{x^4-1}$$

Answer

$$\frac{1}{(x^4-1)} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x+1)(x-1)(1+x^2)}$$

$$\text{Let } \frac{1}{(x+1)(x-1)(1+x^2)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{Cx+D}{(x^2+1)}$$

$$1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x^2-1)$$

$$1 = A(x^3+x-x^2-1) + B(x^3+x+x^2+1) + Cx^3 + Dx^2 - Cx - D$$

$$1 = (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x + (-A+B-D)$$

Equating the coefficient of x^3 , x^2 , x , and constant term, we obtain

$$A+B+C=0$$

$$-A+B+D=0$$

$$A+B-C=0$$

$$-A+B-D=1$$

On solving these equations, we obtain

$$A = -\frac{1}{4}, B = \frac{1}{4}, C = 0, \text{ and } D = -\frac{1}{2}$$

$$\therefore \frac{1}{x^4-1} = \frac{-1}{4(x+1)} + \frac{1}{4(x-1)} - \frac{1}{2(x^2+1)}$$

$$\begin{aligned} \Rightarrow \int \frac{1}{x^4-1} dx &= -\frac{1}{4} \log|x-1| + \frac{1}{4} \log|x+1| - \frac{1}{2} \tan^{-1} x + C \\ &= \frac{1}{4} \log \left| \frac{x+1}{x-1} \right| - \frac{1}{2} \tan^{-1} x + C \end{aligned}$$

Question 16:

$$\frac{1}{x(x^n + 1)} \quad [\text{Hint: multiply numerator and denominator by } x^{n-1} \text{ and put } x^n = t]$$

Answer

$$\frac{1}{x(x^n + 1)}$$

Multiplying numerator and denominator by x^{n-1} , we obtain

$$\frac{1}{x(x^n + 1)} = \frac{x^{n-1}}{x^{n-1}x(x^n + 1)} = \frac{x^{n-1}}{x^n(x^n + 1)}$$

$$\text{Let } x^n = t \Rightarrow x^{n-1} dx = dt$$

$$\therefore \int \frac{1}{x(x^n + 1)} dx = \int \frac{x^{n-1}}{x^n(x^n + 1)} dx = \frac{1}{n} \int \frac{1}{t(t+1)} dt$$

$$\text{Let } \frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{(t+1)}$$

$$1 = A(1+t) + Bt \quad \dots(1)$$

Substituting $t = 0, -1$ in equation (1), we obtain

$$A = 1 \text{ and } B = -1$$

$$\therefore \frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{(t+1)}$$

$$\begin{aligned} \Rightarrow \int \frac{1}{x(x^n + 1)} dx &= \frac{1}{n} \int \left\{ \frac{1}{t} - \frac{1}{(t+1)} \right\} dx \\ &= \frac{1}{n} [\log|t| - \log|t+1|] + C \\ &= -\frac{1}{n} [\log|x^n| - \log|x^n + 1|] + C \\ &= \frac{1}{n} \log \left| \frac{x^n}{x^n + 1} \right| + C \end{aligned}$$

Question 17:

$$\frac{\cos x}{(1 - \sin x)(2 - \sin x)} \quad [\text{Hint: Put } \sin x = t]$$

Answer

$$\frac{\cos x}{(1 - \sin x)(2 - \sin x)}$$

$$\text{Let } \sin x = t \Rightarrow \cos x \, dx = dt$$

$$\therefore \int \frac{\cos x}{(1 - \sin x)(2 - \sin x)} \, dx = \int \frac{dt}{(1 - t)(2 - t)}$$

$$\text{Let } \frac{1}{(1 - t)(2 - t)} = \frac{A}{1 - t} + \frac{B}{2 - t}$$

$$1 = A(2 - t) + B(1 - t) \quad \dots(1)$$

Substituting $t = 2$ and then $t = 1$ in equation (1), we obtain

$A = 1$ and $B = -1$

$$\begin{aligned} \therefore \frac{1}{(1 - t)(2 - t)} &= \frac{1}{1 - t} - \frac{1}{2 - t} \\ \Rightarrow \int \frac{\cos x}{(1 - \sin x)(2 - \sin x)} \, dx &= \int \left[\frac{1}{1 - t} - \frac{1}{2 - t} \right] dt \\ &= -\log|1 - t| + \log|2 - t| + C \\ &= \log \left| \frac{2 - t}{1 - t} \right| + C \\ &= \log \left| \frac{2 - \sin x}{1 - \sin x} \right| + C \end{aligned}$$

Question 18:

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$$

Answer

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \frac{(4x^2+10)}{(x^2+3)(x^2+4)}$$

$$\text{Let } \frac{4x^2+10}{(x^2+3)(x^2+4)} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{x^2+4}$$

$$4x^2+10 = (Ax+B)(x^2+4) + (Cx+D)(x^2+3)$$

$$4x^2+10 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + 3Cx + Dx^2 + 3D$$

$$4x^2+10 = (A+C)x^3 + (B+D)x^2 + (4A+3C)x + (4B+3D)$$

Equating the coefficients of x^3 , x^2 , x , and constant term, we obtain

$$A + C = 0$$

$$B + D = 4$$

$$4A + 3C = 0$$

$$4B + 3D = 10$$

On solving these equations, we obtain

$$A = 0, B = -2, C = 0, \text{ and } D = 6$$

$$\therefore \frac{4x^2+10}{(x^2+3)(x^2+4)} = \frac{-2}{x^2+3} + \frac{6}{x^2+4}$$

$$\begin{aligned}\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} &= 1 - \left(\frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)} \right) \\ \Rightarrow \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx &= \int \left[1 + \frac{2}{(x^2+3)} - \frac{6}{(x^2+4)} \right] dx \\ &= \int \left[1 + \frac{2}{x^2 + (\sqrt{3})^2} - \frac{6}{x^2 + 2^2} \right] dx \\ &= x + 2 \left(\frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} \right) - 6 \left(\frac{1}{2} \tan^{-1} \frac{x}{2} \right) + C \\ &= x + \frac{2}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} - 3 \tan^{-1} \frac{x}{2} + C\end{aligned}$$

Question 19:

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Answer

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Let $x^2 = t \Rightarrow 2x dx = dt$

$$\therefore \int \frac{2x}{(x^2+1)(x^2+3)} dx = \int \frac{dt}{(t+1)(t+3)} \quad \dots(1)$$

$$\text{Let } \frac{1}{(t+1)(t+3)} = \frac{A}{t+1} + \frac{B}{t+3}$$

$$1 = A(t+3) + B(t+1) \quad \dots(1)$$

Substituting $t = -3$ and $t = -1$ in equation (1), we obtain

$$A = \frac{1}{2} \text{ and } B = -\frac{1}{2}$$

$$\therefore \frac{1}{(t+1)(t+3)} = \frac{1}{2(t+1)} - \frac{1}{2(t+3)}$$

$$\begin{aligned} \Rightarrow \int \frac{2x}{(x^2+1)(x^2+3)} dx &= \int \left[\frac{1}{2(t+1)} - \frac{1}{2(t+3)} \right] dt \\ &= \frac{1}{2} \log |(t+1)| - \frac{1}{2} \log |t+3| + C \\ &= \frac{1}{2} \log \left| \frac{t+1}{t+3} \right| + C \\ &= \frac{1}{2} \log \left| \frac{x^2+1}{x^2+3} \right| + C \end{aligned}$$

Question 20:

$$\frac{1}{x(x^4-1)}$$

Answer

$$\frac{1}{x(x^4-1)}$$

Multiplying numerator and denominator by x^3 , we obtain

$$\frac{1}{x(x^4-1)} = \frac{x^3}{x^4(x^4-1)}$$

$$\therefore \int \frac{1}{x(x^4-1)} dx = \int \frac{x^3}{x^4(x^4-1)} dx$$

Let $x^4 = t \Rightarrow 4x^3 dx = dt$

$$\therefore \int \frac{1}{x(x^4-1)} dx = \frac{1}{4} \int \frac{dt}{t(t-1)}$$

$$\text{Let } \frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{(t-1)}$$

$$1 = A(t-1) + Bt \quad \dots(1)$$

Substituting $t = 0$ and 1 in (1), we obtain

$$A = -1 \text{ and } B = 1$$

$$\Rightarrow \frac{1}{t(t+1)} = \frac{-1}{t} + \frac{1}{t-1}$$

$$\begin{aligned} \Rightarrow \int \frac{1}{x(x^4-1)} dx &= \frac{1}{4} \int \left\{ \frac{-1}{t} + \frac{1}{t-1} \right\} dt \\ &= \frac{1}{4} [-\log|t| + \log|t-1|] + C \\ &= \frac{1}{4} \log \left| \frac{t-1}{t} \right| + C \\ &= \frac{1}{4} \log \left| \frac{x^4-1}{x^4} \right| + C \end{aligned}$$

Question 21:

$$\frac{1}{(e^x - 1)} \quad [\text{Hint: Put } e^x = t]$$

Answer

$$\frac{1}{(e^x - 1)}$$

$$\text{Let } e^x = t \Rightarrow e^x dx = dt$$

$$\Rightarrow \int \frac{1}{e^x - 1} dx = \int \frac{1}{t-1} \times \frac{dt}{t} = \int \frac{1}{t(t-1)} dt$$

$$\text{Let } \frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$$

$$1 = A(t-1) + Bt \quad \dots(1)$$

Substituting $t = 1$ and $t = 0$ in equation (1), we obtain

$$A = -1 \text{ and } B = 1$$

$$\therefore \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$$

$$\begin{aligned} \Rightarrow \int \frac{1}{t(t-1)} dt &= \log \left| \frac{t-1}{t} \right| + C \\ &= \log \left| \frac{e^x - 1}{e^x} \right| + C \end{aligned}$$

Question 22:

$$\int \frac{x dx}{(x-1)(x-2)} \text{ equals}$$

A. $\log \left| \frac{(x-1)^2}{x-2} \right| + C$

B. $\log \left| \frac{(x-2)^2}{x-1} \right| + C$

C. $\log \left| \left(\frac{x-1}{x-2} \right)^2 \right| + C$

D. $\log |(x-1)(x-2)| + C$

Answer

$$\text{Let } \frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$x = A(x-2) + B(x-1) \quad \dots(1)$$

Substituting $x = 1$ and 2 in (1), we obtain

$$A = -1 \text{ and } B = 2$$

$$\begin{aligned} \therefore \frac{x}{(x-1)(x-2)} &= -\frac{1}{(x-1)} + \frac{2}{(x-2)} \\ \Rightarrow \int \frac{x}{(x-1)(x-2)} dx &= \int \left\{ \frac{-1}{(x-1)} + \frac{2}{(x-2)} \right\} dx \\ &= -\log|x-1| + 2\log|x-2| + C \\ &= \log \left| \frac{(x-2)^2}{x-1} \right| + C \end{aligned}$$

Hence, the correct Answer is B.

Question 23:

$$\int \frac{dx}{x(x^2+1)} \text{ equals}$$

- A. $\log|x| - \frac{1}{2} \log(x^2+1) + C$
- B. $\log|x| + \frac{1}{2} \log(x^2+1) + C$
- C. $-\log|x| + \frac{1}{2} \log(x^2+1) + C$
- D. $\frac{1}{2} \log|x| + \log(x^2+1) + C$

Answer

$$\text{Let } \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + (Bx+C)x$$

Equating the coefficients of x^2 , x , and constant term, we obtain

$$A + B = 0$$

$$C = 0 \quad A =$$

$$1$$

On solving these equations, we obtain

$$A = 1, B = -1, \text{ and } C = 0$$

$$\begin{aligned}\therefore \frac{1}{x(x^2+1)} &= \frac{1}{x} + \frac{-x}{x^2+1} \\ \Rightarrow \int \frac{1}{x(x^2+1)} dx &= \int \left\{ \frac{1}{x} - \frac{x}{x^2+1} \right\} dx \\ &= \log|x| - \frac{1}{2} \log|x^2+1| + C\end{aligned}$$

Hence, the correct Answer is A.

Exercise 7.6

Question 1:

$x \sin x$

Answer

$$\text{Let } I = \int x \sin x \, dx$$

Taking x as first function and $\sin x$ as second function and integrating by parts, we obtain

$$\begin{aligned} I &= x \int \sin x \, dx - \int \left\{ \left(\frac{d}{dx} x \right) \int \sin x \, dx \right\} dx \\ &= x(-\cos x) - \int 1 \cdot (-\cos x) \, dx \\ &= -x \cos x + \sin x + C \end{aligned}$$

Question 2:

$x \sin 3x$

Answer

$$\text{Let } I = \int x \sin 3x \, dx$$

Taking x as first function and $\sin 3x$ as second function and integrating by parts, we obtain

$$\begin{aligned}
 I &= x \int \sin 3x \, dx - \int \left\{ \left(\frac{d}{dx} x \right) \int \sin 3x \, dx \right\} \\
 &= x \left(\frac{-\cos 3x}{3} \right) - \int 1 \cdot \left(\frac{-\cos 3x}{3} \right) dx \\
 &= \frac{-x \cos 3x}{3} + \frac{1}{3} \int \cos 3x \, dx \\
 &= \frac{-x \cos 3x}{3} + \frac{1}{9} \sin 3x + C
 \end{aligned}$$

Question 3:

$$x^2 e^x$$

Answer

$$I = \int x^2 e^x dx$$

Let

Taking x^2 as first function and e^x as second function and integrating by parts, we obtain

$$\begin{aligned}
 I &= x^2 \int e^x dx - \int \left\{ \left(\frac{d}{dx} x^2 \right) \int e^x dx \right\} dx \\
 &= x^2 e^x - \int 2x \cdot e^x dx \\
 &= x^2 e^x - 2 \int x \cdot e^x dx
 \end{aligned}$$

Again integrating by parts, we obtain

$$\begin{aligned}
 &= x^2 e^x - 2 \left[x \cdot \int e^x dx - \int \left\{ \left(\frac{d}{dx} x \right) \cdot \int e^x dx \right\} dx \right] \\
 &= x^2 e^x - 2 \left[x e^x - \int e^x dx \right] \\
 &= x^2 e^x - 2 \left[x e^x - e^x \right] \\
 &= x^2 e^x - 2x e^x + 2e^x + C \\
 &= e^x (x^2 - 2x + 2) + C
 \end{aligned}$$

Question 4:

$$x \log x$$

Answer

Let $I = \int x \log x dx$

Taking log x as first function and x as second function and integrating by parts, we obtain

$$\begin{aligned} I &= \log x \int x dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x dx \right\} dx \\ &= \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \\ &= \frac{x^2 \log x}{2} - \int \frac{x}{2} dx \\ &= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C \end{aligned}$$

Question 5: x

log 2x

Answer

Let $I = \int x \log 2x dx$

Taking log 2x as first function and x as second function and integrating by parts, we obtain

$$\begin{aligned} I &= \log 2x \int x dx - \int \left\{ \left(\frac{d}{dx} 2 \log x \right) \int x dx \right\} dx \\ &= \log 2x \cdot \frac{x^2}{2} - \int \frac{2}{2x} \cdot \frac{x^2}{2} dx \\ &= \frac{x^2 \log 2x}{2} - \int \frac{x}{2} dx \\ &= \frac{x^2 \log 2x}{2} - \frac{x^2}{4} + C \end{aligned}$$

Question 6:

$x^2 \log x$ Answer

Let $I = \int x^2 \log x \, dx$

Taking $\log x$ as first function and x^2 as second function and integrating by parts, we obtain

$$\begin{aligned} I &= \log x \int x^2 \, dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x^2 \, dx \right\} dx \\ &= \log x \left(\frac{x^3}{3} \right) - \int \frac{1}{x} \cdot \frac{x^3}{3} \, dx \\ &= \frac{x^3 \log x}{3} - \int \frac{x^2}{3} \, dx \\ &= \frac{x^3 \log x}{3} - \frac{x^3}{9} + C \end{aligned}$$

Question 7:

$$x \sin^{-1} x$$

Answer

$$I = \int x \sin^{-1} x \, dx$$

Let

Taking $\sin^{-1} x$ as first function and x as second function and integrating by parts, we obtain

$$\begin{aligned} I &= \sin^{-1} x \int x \, dx - \int \left\{ \left(\frac{d}{dx} \sin^{-1} x \right) \int x \, dx \right\} dx \\ &= \sin^{-1} x \left(\frac{x^2}{2} \right) - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx \\ &= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} dx \\ &= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left\{ \frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \right\} dx \\ &= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left\{ \sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}} \right\} dx \\ &= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left\{ \int \sqrt{1-x^2} dx - \int \frac{1}{\sqrt{1-x^2}} dx \right\} \\ &= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left\{ \frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x - \sin^{-1} x \right\} + C \\ &= \frac{x^2 \sin^{-1} x}{2} + \frac{x}{4} \sqrt{1-x^2} + \frac{1}{4} \sin^{-1} x - \frac{1}{2} \sin^{-1} x + C \\ &= \frac{1}{4} (2x^2 - 1) \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} + C \end{aligned}$$

Question 8:

$$x \tan^{-1} x$$

Answer

$$I = \int x \tan^{-1} x \, dx$$

Let

Taking $\tan^{-1} x$ as first function and x as second function and integrating by parts, we obtain

$$\begin{aligned} I &= \tan^{-1} x \int x \, dx - \int \left\{ \left(\frac{d}{dx} \tan^{-1} x \right) \int x \, dx \right\} dx \\ &= \tan^{-1} x \left(\frac{x^2}{2} \right) - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} dx \\ &= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \frac{x^2}{1+x^2} dx \\ &= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} \right) dx \\ &= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx \\ &= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} (x - \tan^{-1} x) + C \\ &= \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + C \end{aligned}$$

Question 9:

$$x \cos^{-1} x$$

Answer

$$I = \int x \cos^{-1} x \, dx$$

Let

Taking $\cos^{-1} x$ as first function and x as second function and integrating by parts, we obtain

$$\begin{aligned}
 I &= \cos^{-1} x \int x dx - \int \left\{ \left(\frac{d}{dx} \cos^{-1} x \right) \int x dx \right\} dx \\
 &= \cos^{-1} x \frac{x^2}{2} - \int \frac{-1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx \\
 &= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx \\
 &= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \left\{ \sqrt{1-x^2} + \left(\frac{-1}{\sqrt{1-x^2}} \right) \right\} dx \\
 &= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \sqrt{1-x^2} dx - \frac{1}{2} \int \left(\frac{-1}{\sqrt{1-x^2}} \right) dx \\
 &= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} I_1 - \frac{1}{2} \cos^{-1} x \qquad \dots(1)
 \end{aligned}$$

where, $I_1 = \int \sqrt{1-x^2} dx$

$$\begin{aligned}
 \Rightarrow I_1 &= x\sqrt{1-x^2} - \int \frac{d}{dx} \sqrt{1-x^2} \int x dx \\
 \Rightarrow I_1 &= x\sqrt{1-x^2} - \int \frac{-2x}{2\sqrt{1-x^2}} \cdot x dx \\
 \Rightarrow I_1 &= x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}} dx \\
 \Rightarrow I_1 &= x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx \\
 \Rightarrow I_1 &= x\sqrt{1-x^2} - \left\{ \int \sqrt{1-x^2} dx + \int \frac{-dx}{\sqrt{1-x^2}} \right\} \\
 \Rightarrow I_1 &= x\sqrt{1-x^2} - \{I_1 + \cos^{-1} x\} \\
 \Rightarrow 2I_1 &= x\sqrt{1-x^2} - \cos^{-1} x \\
 \therefore I_1 &= \frac{x}{2} \sqrt{1-x^2} - \frac{1}{2} \cos^{-1} x
 \end{aligned}$$

Substituting in (1), we obtain

$$\begin{aligned}
 I &= \frac{x \cos^{-1} x}{2} - \frac{1}{2} \left(\frac{x}{2} \sqrt{1-x^2} - \frac{1}{2} \cos^{-1} x \right) - \frac{1}{2} \cos^{-1} x \\
 &= \frac{(2x^2-1)}{4} \cos^{-1} x - \frac{x}{4} \sqrt{1-x^2} + C
 \end{aligned}$$

Question 10:

$$\left(\sin^{-1} x\right)^2$$

Answer

$$I = \int \left(\sin^{-1} x\right)^2 \cdot 1 \, dx$$

Let

Taking $\left(\sin^{-1} x\right)^2$ as first function and 1 as second function and integrating by parts, we obtain

$$\begin{aligned} I &= \left(\sin^{-1} x\right) \int 1 \, dx - \int \left\{ \frac{d}{dx} \left(\sin^{-1} x\right)^2 \cdot \int 1 \cdot dx \right\} dx \\ &= \left(\sin^{-1} x\right)^2 \cdot x - \int \frac{2 \sin^{-1} x}{\sqrt{1-x^2}} \cdot x \, dx \\ &= x \left(\sin^{-1} x\right)^2 + \int \sin^{-1} x \cdot \left(\frac{-2x}{\sqrt{1-x^2}} \right) dx \\ &= x \left(\sin^{-1} x\right)^2 + \left[\sin^{-1} x \int \frac{-2x}{\sqrt{1-x^2}} dx - \int \left\{ \left(\frac{d}{dx} \sin^{-1} x \right) \int \frac{-2x}{\sqrt{1-x^2}} dx \right\} dx \right] \\ &= x \left(\sin^{-1} x\right)^2 + \left[\sin^{-1} x \cdot 2\sqrt{1-x^2} - \int \frac{1}{\sqrt{1-x^2}} \cdot 2\sqrt{1-x^2} dx \right] \\ &= x \left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2} \sin^{-1} x - \int 2 \, dx \\ &= x \left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2} \sin^{-1} x - 2x + C \end{aligned}$$

Question 11:

$$\frac{x \cos^{-1} x}{\sqrt{1-x^2}}$$

Answer

$$I = \int \frac{x \cos^{-1} x}{\sqrt{1-x^2}} dx$$

Let

$$I = \frac{-1}{2} \int \frac{-2x}{\sqrt{1-x^2}} \cdot \cos^{-1} x dx$$

Taking $\cos^{-1} x$ as first function and $\left(\frac{-2x}{\sqrt{1-x^2}}\right)$ as second function and integrating by parts, we obtain

$$\begin{aligned} I &= \frac{-1}{2} \left[\cos^{-1} x \int \frac{-2x}{\sqrt{1-x^2}} dx - \int \left\{ \left(\frac{d}{dx} \cos^{-1} x \right) \int \frac{-2x}{\sqrt{1-x^2}} dx \right\} dx \right] \\ &= \frac{-1}{2} \left[\cos^{-1} x \cdot 2\sqrt{1-x^2} - \int \frac{-1}{\sqrt{1-x^2}} \cdot 2\sqrt{1-x^2} dx \right] \\ &= \frac{-1}{2} \left[2\sqrt{1-x^2} \cos^{-1} x + \int 2 dx \right] \\ &= \frac{-1}{2} \left[2\sqrt{1-x^2} \cos^{-1} x + 2x \right] + C \\ &= - \left[\sqrt{1-x^2} \cos^{-1} x + x \right] + C \end{aligned}$$

Question 12:

$$x \sec^2 x$$

Answer

$$I = \int x \sec^2 x dx$$

Let

Taking x as first function and $\sec^2 x$ as second function and integrating by parts, we obtain

$$\begin{aligned} I &= x \int \sec^2 x dx - \int \left\{ \left(\frac{d}{dx} x \right) \int \sec^2 x dx \right\} dx \\ &= x \tan x - \int 1 \cdot \tan x dx \\ &= x \tan x + \log |\cos x| + C \end{aligned}$$

Question 13:

$$\tan^{-1} x$$

Answer

$$I = \int 1 \cdot \tan^{-1} x dx$$

Let

Taking $\tan^{-1} x$ as first function and 1 as second function and integrating by parts, we obtain

$$\begin{aligned} I &= \tan^{-1} x \int 1 dx - \int \left\{ \left(\frac{d}{dx} \tan^{-1} x \right) \int 1 \cdot dx \right\} dx \\ &= \tan^{-1} x \cdot x - \int \frac{1}{1+x^2} \cdot x dx \\ &= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx \\ &= x \tan^{-1} x - \frac{1}{2} \log |1+x^2| + C \\ &= x \tan^{-1} x - \frac{1}{2} \log(1+x^2) + C \end{aligned}$$

Question 14:

$$x(\log x)^2$$

Answer

$$I = \int x(\log x)^2 dx$$

Taking $(\log x)^2$ as first function and 1 as second function and integrating by parts, we obtain

$$\begin{aligned} I &= (\log x)^2 \int x dx - \int \left[\left(\frac{d}{dx} \log x \right)^2 \int x dx \right] dx \\ &= \frac{x^2}{2} (\log x)^2 - \left[\int 2 \log x \cdot \frac{1}{x} \cdot \frac{x^2}{2} dx \right] \\ &= \frac{x^2}{2} (\log x)^2 - \int x \log x dx \end{aligned}$$

Again integrating by parts, we obtain

$$\begin{aligned} I &= \frac{x^2}{2} (\log x)^2 - \left[\log x \int x dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x dx \right\} dx \right] \\ &= \frac{x^2}{2} (\log x)^2 - \left[\frac{x^2}{2} - \log x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \right] \\ &= \frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \log x + \frac{1}{2} \int x dx \\ &= \frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \log x + \frac{x^2}{4} + C \end{aligned}$$

Question 15:

$$(x^2 + 1) \log x$$

Answer

$$I = \int (x^2 + 1) \log x \, dx = \int x^2 \log x \, dx + \int \log x \, dx$$

Let

$$\text{Let } I = I_1 + I_2 \dots (1)$$

$$\text{Where, } I_1 = \int x^2 \log x \, dx \quad \text{and} \quad I_2 = \int \log x \, dx$$

$$I_1 = \int x^2 \log x \, dx$$

Taking $\log x$ as first function and x^2 as second function and integrating by parts, we obtain

$$\begin{aligned} I_1 &= \log x \cdot \int x^2 \, dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x^2 \, dx \right\} dx \\ &= \log x \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} \, dx \\ &= \frac{x^3}{3} \log x - \frac{1}{3} \left(\int x^2 \, dx \right) \\ &= \frac{x^3}{3} \log x - \frac{x^3}{9} + C_1 \quad \dots (2) \end{aligned}$$

$$I_2 = \int \log x \, dx$$

Taking $\log x$ as first function and 1 as second function and integrating by parts, we obtain

$$\begin{aligned}
 I_2 &= \log x \int 1 \cdot dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int 1 \cdot dx \right\} \\
 &= \log x \cdot x - \int \frac{1}{x} \cdot x dx \\
 &= x \log x - \int 1 dx \\
 &= x \log x - x + C_2 \qquad \dots (3)
 \end{aligned}$$

Using equations (2) and (3) in (1), we obtain

$$\begin{aligned}
 I &= \frac{x^3}{3} \log x - \frac{x^3}{9} + C_1 + x \log x - x + C_2 \\
 &= \frac{x^3}{3} \log x - \frac{x^3}{9} + x \log x - x + (C_1 + C_2) \\
 &= \left(\frac{x^3}{3} + x \right) \log x - \frac{x^3}{9} - x + C
 \end{aligned}$$

Question 16:

$$e^x (\sin x + \cos x)$$

Answer

$$I = \int e^x (\sin x + \cos x) dx$$

Let

$$f(x) = \sin x$$

Let

$$f'(x) = \cos x$$

⇒

$$I = \int e^x \{f(x) + f'(x)\} dx$$

⇒

$$\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$$

It is known that,

$$\therefore I = e^x \sin x + C$$

Question 17:

$$\frac{xe^x}{(1+x)^2}$$

Answer

$$I = \int \frac{xe^x}{(1+x)^2} dx = \int e^x \left\{ \frac{x}{(1+x)^2} \right\} dx$$

Let

$$= \int e^x \left\{ \frac{1+x-1}{(1+x)^2} \right\} dx$$

$$= \int e^x \left\{ \frac{1}{1+x} - \frac{1}{(1+x)^2} \right\} dx$$

Let $f(x) = \frac{1}{1+x} \Rightarrow f'(x) = \frac{-1}{(1+x)^2}$

$$\Rightarrow \int \frac{xe^x}{(1+x)^2} dx = \int e^x \{f(x) + f'(x)\} dx$$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

$$\therefore \int \frac{xe^x}{(1+x)^2} dx = \frac{e^x}{1+x} + C$$

Question 18:

$$e^x \left(\frac{1 + \sin x}{1 + \cos x} \right)$$

Answer

$$\begin{aligned}
 & e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) \\
 &= e^x \left(\frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right) \\
 &= \frac{e^x \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right)^2}{2 \cos^2 \frac{x}{2}} \\
 &= \frac{1}{2} e^x \cdot \left(\frac{\sin \frac{x}{2} + \cos \frac{x}{2}}{\cos \frac{x}{2}} \right)^2 \\
 &= \frac{1}{2} e^x \left[\tan \frac{x}{2} + 1 \right]^2 \\
 &= \frac{1}{2} e^x \left(1 + \tan \frac{x}{2} \right)^2 \\
 &= \frac{1}{2} e^x \left[1 + \tan^2 \frac{x}{2} + 2 \tan \frac{x}{2} \right] \\
 &= \frac{1}{2} e^x \left[\sec^2 \frac{x}{2} + 2 \tan \frac{x}{2} \right] \\
 & \frac{e^x (1 + \sin x) dx}{(1 + \cos x)} = e^x \left[\frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right] \quad \dots(1)
 \end{aligned}$$

Let $\tan \frac{x}{2} = f(x) \Rightarrow f'(x) = \frac{1}{2} \sec^2 \frac{x}{2}$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

From equation (1), we obtain

$$\int \frac{e^x (1 + \sin x)}{(1 + \cos x)} dx = e^x \tan \frac{x}{2} + C$$

Question 19:

$$e^x \left(\frac{1}{x} - \frac{1}{x^2} \right)$$

Answer

$$\text{Let } I = \int e^x \left[\frac{1}{x} - \frac{1}{x^2} \right] dx$$

$$\text{Also, let } \frac{1}{x} = f(x) \Rightarrow f'(x) = \frac{-1}{x^2}$$

$$\text{It is known that, } \int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$$

$$\therefore I = \frac{e^x}{x} + C$$

Question 20:

$$\frac{(x-3)e^x}{(x-1)^3}$$

Answer

$$\begin{aligned} \int e^x \left\{ \frac{x-3}{(x-1)^3} \right\} dx &= \int e^x \left\{ \frac{x-1-2}{(x-1)^3} \right\} dx \\ &= \int e^x \left\{ \frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right\} dx \end{aligned}$$

$$\text{Let } f(x) = \frac{1}{(x-1)^2} \Rightarrow f'(x) = \frac{-2}{(x-1)^3}$$

$$\text{It is known that, } \int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$$

$$\therefore \int e^x \left\{ \frac{(x-3)}{(x-1)^2} \right\} dx = \frac{e^x}{(x-1)^2} + C$$

Question 21:

$$e^{2x} \sin x$$

Answer

$$I = \int e^{2x} \sin x \, dx \quad \dots(1)$$

Let

Integrating by parts, we obtain

$$I = \sin x \int e^{2x} \, dx - \int \left\{ \left(\frac{d}{dx} \sin x \right) \int e^{2x} \, dx \right\} dx$$

$$\Rightarrow I = \sin x \cdot \frac{e^{2x}}{2} - \int \cos x \cdot \frac{e^{2x}}{2} \, dx$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \int e^{2x} \cos x \, dx$$

Again integrating by parts, we obtain

$$I = \frac{e^{2x} \cdot \sin x}{2} - \frac{1}{2} \left[\cos x \int e^{2x} \, dx - \int \left\{ \left(\frac{d}{dx} \cos x \right) \int e^{2x} \, dx \right\} dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \left[\cos x \cdot \frac{e^{2x}}{2} - \int (-\sin x) \frac{e^{2x}}{2} \, dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \cdot \sin x}{2} - \frac{1}{2} \left[\frac{e^{2x} \cos x}{2} + \frac{1}{2} \int e^{2x} \sin x \, dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{4} I \quad \text{[From (1)]}$$

$$\Rightarrow I + \frac{1}{4} I = \frac{e^{2x} \cdot \sin x}{2} - \frac{e^{2x} \cos x}{4}$$

$$\Rightarrow \frac{5}{4} I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4}$$

$$\Rightarrow I = \frac{4}{5} \left[\frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} \right] + C$$

$$\Rightarrow I = \frac{e^{2x}}{5} [2 \sin x - \cos x] + C$$

Question 22:

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Answer

$$\begin{aligned} \text{Let } x = \tan \theta &\Rightarrow dx = \sec^2 \theta d\theta \\ \therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right) &= \sin^{-1}\left(\frac{2 \tan \theta}{1 + \tan^2 \theta}\right) = \sin^{-1}(\sin 2\theta) \\ &= 2\theta \end{aligned}$$

$$\Rightarrow \int \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx = \int 2\theta \cdot \sec^2 \theta d\theta = 2 \int \theta \cdot \sec^2 \theta d\theta$$

Integrating by parts, we obtain

$$\begin{aligned} &2 \left[\theta \cdot \int \sec^2 \theta d\theta - \int \left\{ \left(\frac{d}{d\theta} \theta \right) \int \sec^2 \theta d\theta \right\} d\theta \right] \\ &= 2 \left[\theta \cdot \tan \theta - \int \tan \theta d\theta \right] \\ &= 2 \left[\theta \tan \theta + \log |\cos \theta| \right] + C \\ &= 2 \left[x \tan^{-1} x + \log \left| \frac{1}{\sqrt{1+x^2}} \right| \right] + C \\ &= 2x \tan^{-1} x + 2 \log (1+x^2)^{-\frac{1}{2}} + C \\ &= 2x \tan^{-1} x + 2 \left[-\frac{1}{2} \log (1+x^2) \right] + C \\ &= 2x \tan^{-1} x - \log (1+x^2) + C \end{aligned}$$

Question 23:

$\int x^2 e^{x^3} dx$ equals

(A) $\frac{1}{3} e^{x^3} + C$

(B) $\frac{1}{3} e^{x^2} + C$

(C) $\frac{1}{2} e^{x^3} + C$

(D) $\frac{1}{3} e^{x^2} + C$

Answer

$I = \int x^2 e^{x^3} dx$

Let

Also, let $x^3 = t \Rightarrow 3x^2 dx = dt$

$$\begin{aligned} \Rightarrow I &= \frac{1}{3} \int e^t dt \\ &= \frac{1}{3} (e^t) + C \\ &= \frac{1}{3} e^{x^3} + C \end{aligned}$$

Hence, the correct Answer is A.

Question 24:

$$\int e^x \sec x (1 + \tan x) dx \text{ equals}$$

- (A) $e^x \cos x + C$ (B) $e^x \sec x + C$
(C) $e^x \sin x + C$ (D) $e^x \tan x + C$

Answer

$$\int e^x \sec x (1 + \tan x) dx$$

$$\text{Let } I = \int e^x \sec x (1 + \tan x) dx = \int e^x (\sec x + \sec x \tan x) dx$$

$$\text{Also, let } \sec x = f(x) \Rightarrow \sec x \tan x = f'(x)$$

$$\text{It is known that, } \int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$$

$$\therefore I = e^x \sec x + C$$

Hence, the correct Answer is B.

Exercise 7.7

Question 1:

$$\sqrt{4-x^2}$$

Answer

$$\text{Let } I = \int \sqrt{4-x^2} dx = \int \sqrt{(2)^2 - (x)^2} dx$$

$$\text{It is known that, } \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$$

$$\begin{aligned} \therefore I &= \frac{x}{2} \sqrt{4-x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} + C \\ &= \frac{x}{2} \sqrt{4-x^2} + 2 \sin^{-1} \frac{x}{2} + C \end{aligned}$$

Question 2:

$$\sqrt{1-4x^2}$$

Answer

$$\text{Let } I = \int \sqrt{1-4x^2} dx = \int \sqrt{(1)^2 - (2x)^2} dx$$

$$\text{Let } 2x = t \Rightarrow 2 dx = dt$$

$$\therefore I = \frac{1}{2} \int \sqrt{(1)^2 - (t)^2} dt$$

$$\text{It is known that, } \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$$

$$\begin{aligned} \Rightarrow I &= \frac{1}{2} \left[\frac{t}{2} \sqrt{1-t^2} + \frac{1}{2} \sin^{-1} t \right] + C \\ &= \frac{t}{4} \sqrt{1-t^2} + \frac{1}{4} \sin^{-1} t + C \\ &= \frac{2x}{4} \sqrt{1-4x^2} + \frac{1}{4} \sin^{-1} 2x + C \\ &= \frac{x}{2} \sqrt{1-4x^2} + \frac{1}{4} \sin^{-1} 2x + C \end{aligned}$$

Question 3:

$$\sqrt{x^2 + 4x + 6}$$

Answer

$$\begin{aligned} \text{Let } I &= \int \sqrt{x^2 + 4x + 6} \, dx \\ &= \int \sqrt{x^2 + 4x + 4 + 2} \, dx \\ &= \int \sqrt{(x^2 + 4x + 4) + 2} \, dx \\ &= \int \sqrt{(x+2)^2 + (\sqrt{2})^2} \, dx \end{aligned}$$

It is known that, $\int \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + C$

$$\begin{aligned} \therefore I &= \frac{(x+2)}{2} \sqrt{x^2 + 4x + 6} + \frac{2}{2} \log |(x+2) + \sqrt{x^2 + 4x + 6}| + C \\ &= \frac{(x+2)}{2} \sqrt{x^2 + 4x + 6} + \log |(x+2) + \sqrt{x^2 + 4x + 6}| + C \end{aligned}$$

Question 4:

$$\sqrt{x^2 + 4x + 1}$$

Answer

$$\begin{aligned} \text{Let } I &= \int \sqrt{x^2 + 4x + 1} \, dx \\ &= \int \sqrt{(x^2 + 4x + 4) - 3} \, dx \\ &= \int \sqrt{(x+2)^2 - (\sqrt{3})^2} \, dx \end{aligned}$$

It is known that, $\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + C$

$$\therefore I = \frac{(x+2)}{2} \sqrt{x^2 + 4x + 1} - \frac{3}{2} \log |(x+2) + \sqrt{x^2 + 4x + 1}| + C$$

Question 5:

$$\sqrt{1-4x-x^2}$$

Answer

$$\begin{aligned} \text{Let } I &= \int \sqrt{1-4x-x^2} \, dx \\ &= \int \sqrt{1-(x^2+4x+4-4)} \, dx \\ &= \int \sqrt{1+4-(x+2)^2} \, dx \\ &= \int \sqrt{(\sqrt{5})^2-(x+2)^2} \, dx \end{aligned}$$

It is known that, $\int \sqrt{a^2-x^2} \, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

$$\therefore I = \frac{(x+2)}{2}\sqrt{1-4x-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + C$$

Question 6:

$$\sqrt{x^2+4x-5}$$

Answer

$$\begin{aligned} \text{Let } I &= \int \sqrt{x^2+4x-5} \, dx \\ &= \int \sqrt{(x^2+4x+4)-9} \, dx \\ &= \int \sqrt{(x+2)^2-(3)^2} \, dx \end{aligned}$$

It is known that, $\int \sqrt{x^2-a^2} \, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + C$

$$\therefore I = \frac{(x+2)}{2}\sqrt{x^2+4x-5} - \frac{9}{2}\log|(x+2)+\sqrt{x^2+4x-5}| + C$$

Question 7:

$$\sqrt{1+3x-x^2}$$

Answer

$$\begin{aligned} \text{Let } I &= \int \sqrt{1+3x-x^2} \, dx \\ &= \int \sqrt{1 - \left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right)} \, dx \\ &= \int \sqrt{\left(1 + \frac{9}{4}\right) - \left(x - \frac{3}{2}\right)^2} \, dx \\ &= \int \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2} \, dx \end{aligned}$$

It is known that, $\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$

$$\begin{aligned} \therefore I &= \frac{x - \frac{3}{2}}{2} \sqrt{1+3x-x^2} + \frac{13}{4 \times 2} \sin^{-1} \left(\frac{x - \frac{3}{2}}{\frac{\sqrt{13}}{2}} \right) + C \\ &= \frac{2x - 3}{4} \sqrt{1+3x-x^2} + \frac{13}{8} \sin^{-1} \left(\frac{2x - 3}{\sqrt{13}} \right) + C \end{aligned}$$

Question 8:

$$\sqrt{x^2 + 3x}$$

Answer

$$\begin{aligned} \text{Let } I &= \int \sqrt{x^2 + 3x} \, dx \\ &= \int \sqrt{x^2 + 3x + \frac{9}{4} - \frac{9}{4}} \, dx \\ &= \int \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2} \, dx \end{aligned}$$

It is known that, $\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$

$$\begin{aligned} \therefore I &= \frac{\left(x + \frac{3}{2}\right)}{2} \sqrt{x^2 + 3x} - \frac{9}{2} \log \left| \left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x} \right| + C \\ &= \frac{(2x+3)}{4} \sqrt{x^2 + 3x} - \frac{9}{8} \log \left| \left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x} \right| + C \end{aligned}$$

Question 9:

$$\sqrt{1 + \frac{x^2}{9}}$$

Answer

$$\text{Let } I = \int \sqrt{1 + \frac{x^2}{9}} dx = \frac{1}{3} \int \sqrt{9 + x^2} dx = \frac{1}{3} \int \sqrt{(3)^2 + x^2} dx$$

$$\text{It is known that, } \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + C$$

$$\begin{aligned} \therefore I &= \frac{1}{3} \left[\frac{x}{2} \sqrt{x^2 + 9} + \frac{9}{2} \log |x + \sqrt{x^2 + 9}| \right] + C \\ &= \frac{x}{6} \sqrt{x^2 + 9} + \frac{3}{2} \log |x + \sqrt{x^2 + 9}| + C \end{aligned}$$

Question 10:

$$\int \sqrt{1 + x^2} dx \text{ is equal to}$$

A. $\frac{x}{2} \sqrt{1 + x^2} + \frac{1}{2} \log |x + \sqrt{1 + x^2}| + C$

B. $\frac{2}{3} (1 + x^2)^{\frac{2}{3}} + C$

C. $\frac{2}{3} x (1 + x^2)^{\frac{3}{2}} + C$

D. $\frac{x^2}{2}\sqrt{1+x^2} + \frac{1}{2}x^2 \log|x + \sqrt{1+x^2}| + C$

Answer

It is known that, $\int \sqrt{a^2 + x^2} dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \log|x + \sqrt{x^2 + a^2}| + C$

$\therefore \int \sqrt{1+x^2} dx = \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2} \log|x + \sqrt{1+x^2}| + C$

Hence, the correct Answer is A.

Question 11:

$\int \sqrt{x^2 - 8x + 7} dx$ is equal to

- A. $\frac{1}{2}(x-4)\sqrt{x^2 - 8x + 7} + 9 \log|x - 4 + \sqrt{x^2 - 8x + 7}| + C$
- B. $\frac{1}{2}(x+4)\sqrt{x^2 - 8x + 7} + 9 \log|x + 4 + \sqrt{x^2 - 8x + 7}| + C$
- C. $\frac{1}{2}(x-4)\sqrt{x^2 - 8x + 7} - 3\sqrt{2} \log|x - 4 + \sqrt{x^2 - 8x + 7}| + C$
- D. $\frac{1}{2}(x-4)\sqrt{x^2 - 8x + 7} - \frac{9}{2} \log|x - 4 + \sqrt{x^2 - 8x + 7}| + C$

Answer

$$\begin{aligned} \text{Let } I &= \int \sqrt{x^2 - 8x + 7} dx \\ &= \int \sqrt{(x^2 - 8x + 16) - 9} dx \\ &= \int \sqrt{(x-4)^2 - (3)^2} dx \end{aligned}$$

It is known that, $\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log|x + \sqrt{x^2 - a^2}| + C$

$$\therefore I = \frac{(x-4)}{2} \sqrt{x^2 - 8x + 7} - \frac{9}{2} \log|(x-4) + \sqrt{x^2 - 8x + 7}| + C$$

Hence, the correct Answer is D.

Exercise 7.8 Question

1:

$$\int_0^b x dx$$

Answer

It is known that,

$$\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + \dots + f(a+(n-1)h)], \text{ where } h = \frac{b-a}{n}$$

Here, $a = a$, $b = b$, and $f(x) = x$

$$\begin{aligned} \therefore \int_a^b x dx &= (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [a + (a+h) \dots (a+2h) \dots a + (n-1)h] \\ &= (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} \left[\underbrace{(a+a+a+\dots+a)}_{n \text{ times}} + (h+2h+3h+\dots+(n-1)h) \right] \\ &= (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [na + h(1+2+3+\dots+(n-1))] \\ &= (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} \left[na + h \left\{ \frac{(n-1)(n)}{2} \right\} \right] \\ &= (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} \left[na + \frac{n(n-1)h}{2} \right] \\ &= (b-a) \lim_{n \rightarrow \infty} \frac{n}{n} \left[a + \frac{(n-1)h}{2} \right] \\ &= (b-a) \lim_{n \rightarrow \infty} \left[a + \frac{(n-1)h}{2} \right] \\ &= (b-a) \lim_{n \rightarrow \infty} \left[a + \frac{(n-1)(b-a)}{2n} \right] \\ &= (b-a) \lim_{n \rightarrow \infty} \left[a + \frac{\left(1 - \frac{1}{n}\right)(b-a)}{2} \right] \\ &= (b-a) \left[a + \frac{(b-a)}{2} \right] \\ &= (b-a) \left[\frac{2a+b-a}{2} \right] \\ &= \frac{(b-a)(b+a)}{2} \\ &= \frac{1}{2}(b^2 - a^2) \end{aligned}$$

Question 2:

$$\int_0^5 (x+1) dx$$

Answer

$$\text{Let } I = \int_0^5 (x+1) dx$$

It is known that,

$$\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) \dots f(a+(n-1)h)], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 0$, $b = 5$, and $f(x) = (x+1)$

$$\Rightarrow h = \frac{5-0}{n} = \frac{5}{n}$$

$$\begin{aligned} \therefore \int_0^5 (x+1) dx &= (5-0) \lim_{n \rightarrow \infty} \frac{1}{n} \left[f(0) + f\left(\frac{5}{n}\right) + \dots + f\left((n-1)\frac{5}{n}\right) \right] \\ &= 5 \lim_{n \rightarrow \infty} \frac{1}{n} \left[1 + \left(\frac{5}{n} + 1\right) + \dots + \left\{ 1 + \left(\frac{5(n-1)}{n}\right) \right\} \right] \\ &= 5 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(1 + 1 + 1 \dots 1\right) + \left[\frac{5}{n} + 2 \cdot \frac{5}{n} + 3 \cdot \frac{5}{n} + \dots + (n-1) \frac{5}{n}\right] \right] \\ &= 5 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{5}{n} \{1 + 2 + 3 \dots (n-1)\} \right] \\ &= 5 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{5}{n} \cdot \frac{(n-1)n}{2} \right] \\ &= 5 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{5(n-1)}{2} \right] \\ &= 5 \lim_{n \rightarrow \infty} \left[1 + \frac{5}{2} \left(1 - \frac{1}{n}\right) \right] \\ &= 5 \left[1 + \frac{5}{2} \right] \\ &= 5 \left[\frac{7}{2} \right] \\ &= \frac{35}{2} \end{aligned}$$

Question 3:

$$\int_2^3 x^2 dx$$

Answer

It is known that,

$$\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + f(a+2h) \dots f\{a+(n-1)h\}], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 2$, $b = 3$, and $f(x) = x^2$

$$\Rightarrow h = \frac{3-2}{n} = \frac{1}{n}$$

$$\begin{aligned} \therefore \int_2^3 x^2 dx &= (3-2) \lim_{n \rightarrow \infty} \frac{1}{n} \left[f(2) + f\left(2 + \frac{1}{n}\right) + f\left(2 + \frac{2}{n}\right) \dots f\left\{2 + (n-1)\frac{1}{n}\right\} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[(2)^2 + \left(2 + \frac{1}{n}\right)^2 + \left(2 + \frac{2}{n}\right)^2 + \dots \left(2 + \frac{(n-1)}{n}\right)^2 \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[2^2 + \left\{ 2^2 + \left(\frac{1}{n}\right)^2 + 2 \cdot 2 \cdot \frac{1}{n} \right\} + \dots + \left\{ (2)^2 + \frac{(n-1)^2}{n^2} + 2 \cdot 2 \cdot \frac{(n-1)}{n} \right\} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(2^2 + \dots + 2^2 \right) + \left\{ \left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \dots + \left(\frac{n-1}{n}\right)^2 \right\} + 2 \cdot 2 \cdot \left\{ \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \dots + \frac{(n-1)}{n} \right\} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[4n + \frac{1}{n^2} \{1^2 + 2^2 + 3^2 \dots + (n-1)^2\} + \frac{4}{n} \{1 + 2 + \dots + (n-1)\} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[4n + \frac{1}{n^2} \left\{ \frac{n(n-1)(2n-1)}{6} \right\} + \frac{4}{n} \left\{ \frac{n(n-1)}{2} \right\} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[4n + \frac{n \left(1 - \frac{1}{n}\right) \left(2 - \frac{1}{n}\right)}{6} + \frac{4n-4}{2} \right] \\ &= \lim_{n \rightarrow \infty} \left[4 + \frac{1}{6} \left(1 - \frac{1}{n}\right) \left(2 - \frac{1}{n}\right) + 2 - \frac{2}{n} \right] \\ &= 4 + \frac{2}{6} + 2 \\ &= \frac{19}{3} \end{aligned}$$

Question 4:

$$\int_1^4 (x^2 - x) dx$$

Answer

$$\begin{aligned} \text{Let } I &= \int_1^4 (x^2 - x) dx \\ &= \int_1^4 x^2 dx - \int_1^4 x dx \end{aligned}$$

$$\text{Let } I = I_1 - I_2, \text{ where } I_1 = \int_1^4 x^2 dx \text{ and } I_2 = \int_1^4 x dx \quad \dots(1)$$

It is known that,

$$\int_a^b f(x) dx = (b - a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + f(a+(n-1)h)], \text{ where } h = \frac{b-a}{n}$$

$$\text{For } I_1 = \int_1^4 x^2 dx,$$

$$a = 1, b = 4, \text{ and } f(x) = x^2$$

$$\therefore h = \frac{4-1}{n} = \frac{3}{n}$$

$$\begin{aligned} I_1 &= \int_1^4 x^2 dx = (4-1) \lim_{n \rightarrow \infty} \frac{1}{n} [f(1) + f(1+h) + \dots + f(1+(n-1)h)] \\ &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[1^2 + \left(1 + \frac{3}{n}\right)^2 + \left(1 + 2 \cdot \frac{3}{n}\right)^2 + \dots + \left(1 + \frac{(n-1)3}{n}\right)^2 \right] \\ &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[1^2 + \left\{ 1^2 + \left(\frac{3}{n}\right)^2 + 2 \cdot \frac{3}{n} \right\} + \dots + \left\{ 1^2 + \left(\frac{(n-1)3}{n}\right)^2 + \frac{2 \cdot (n-1) \cdot 3}{n} \right\} \right] \\ &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(1^2 + \dots + 1^2\right) + \left(\frac{3}{n}\right)^2 \left\{ 1^2 + 2^2 + \dots + (n-1)^2 \right\} + 2 \cdot \frac{3}{n} \left\{ 1 + 2 + \dots + (n-1) \right\} \right] \end{aligned}$$

$$\begin{aligned}
 &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{9}{n^2} \left\{ \frac{(n-1)(n)(2n-1)}{6} \right\} + \frac{6}{n} \left\{ \frac{(n-1)(n)}{2} \right\} \right] \\
 &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{9n}{6} \left(1 - \frac{1}{n} \right) \left(2 - \frac{1}{n} \right) + \frac{6n-6}{2} \right] \\
 &= 3 \lim_{n \rightarrow \infty} \left[1 + \frac{9}{6} \left(1 - \frac{1}{n} \right) \left(2 - \frac{1}{n} \right) + 3 - \frac{3}{n} \right] \\
 &= 3[1+3+3] \\
 &= 3[7]
 \end{aligned}$$

$$I_1 = 21 \quad \dots(2)$$

For $I_2 = \int_1^4 x dx$,

$a = 1, b = 4$, and $f(x) = x$

$$\Rightarrow h = \frac{4-1}{n} = \frac{3}{n}$$

$$\begin{aligned}
 \therefore I_2 &= (4-1) \lim_{n \rightarrow \infty} \frac{1}{n} [f(1) + f(1+h) + \dots + f(a+(n-1)h)] \\
 &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} [1 + (1+h) + \dots + (1+(n-1)h)] \\
 &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[1 + \left(1 + \frac{3}{n} \right) + \dots + \left\{ 1 + (n-1) \frac{3}{n} \right\} \right] \\
 &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\underbrace{(1+1+\dots+1)}_{n \text{ times}} + \frac{3}{n} (1+2+\dots+(n-1)) \right] \\
 &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{3}{n} \left\{ \frac{(n-1)n}{2} \right\} \right] \\
 &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[1 + \frac{3}{2} \left(1 - \frac{1}{n} \right) \right] \\
 &= 3 \left[1 + \frac{3}{2} \right] \\
 &= 3 \left[\frac{5}{2} \right]
 \end{aligned}$$

$$I_2 = \frac{15}{2} \quad \dots(3)$$

From equations (2) and (3), we obtain

$$I = I_1 + I_2 = 21 - \frac{15}{2} = \frac{27}{2}$$

Question 5:

$$\int_{-1}^1 e^x dx$$

Answer

$$\text{Let } I = \int_{-1}^1 e^x dx \quad \dots(1)$$

It is known that,

$$\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) \dots f(a+(n-1)h)], \text{ where } h = \frac{b-a}{n}$$

Here, $a = -1$, $b = 1$, and $f(x) = e^x$

$$\therefore h = \frac{1+1}{n} = \frac{2}{n}$$

$$\begin{aligned}
 \therefore I &= (1+1) \lim_{n \rightarrow \infty} \frac{1}{n} \left[f(-1) + f\left(-1 + \frac{2}{n}\right) + f\left(-1 + 2 \cdot \frac{2}{n}\right) + \dots + f\left(-1 + \frac{(n-1)2}{n}\right) \right] \\
 &= 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[e^{-1} + e^{\left(-1 + \frac{2}{n}\right)} + e^{\left(-1 + 2 \cdot \frac{2}{n}\right)} + \dots + e^{\left(-1 + \frac{(n-1)2}{n}\right)} \right] \\
 &= 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[e^{-1} \left\{ 1 + e^{\frac{2}{n}} + e^{\frac{4}{n}} + e^{\frac{6}{n}} + \dots + e^{\frac{(n-1)2}{n}} \right\} \right] \\
 &= 2 \lim_{n \rightarrow \infty} \frac{e^{-1}}{n} \left[\frac{e^{\frac{2n-1}{n}}}{e^{\frac{2-1}{n}}} \right] \\
 &= e^{-1} \times 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{e^2 - 1}{e^{\frac{2}{n} - 1}} \right] \\
 &= \frac{e^{-1} \times 2(e^2 - 1)}{\lim_{\frac{2}{n} \rightarrow 0} \left(\frac{e^{\frac{2}{n}} - 1}{\frac{2}{n}} \right) \times 2} \\
 &= e^{-1} \left[\frac{2(e^2 - 1)}{2} \right] \qquad \left[\lim_{h \rightarrow 0} \left(\frac{e^h - 1}{h} \right) = 1 \right] \\
 &= \frac{e^2 - 1}{e} \\
 &= \left(e - \frac{1}{e} \right)
 \end{aligned}$$

Question 6:

$$\int_0^4 (x + e^{2x}) dx$$

Answer

It is known that,

$$\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + \dots + f(a+(n-1)h)], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 0$, $b = 4$, and $f(x) = x + e^{2x}$

$$\therefore h = \frac{4-0}{n} = \frac{4}{n}$$

$$\begin{aligned} \Rightarrow \int_0^4 (x + e^{2x}) dx &= (4-0) \lim_{n \rightarrow \infty} \frac{1}{n} [f(0) + f(h) + f(2h) + \dots + f((n-1)h)] \\ &= 4 \lim_{n \rightarrow \infty} \frac{1}{n} [(0 + e^0) + (h + e^{2h}) + (2h + e^{2 \cdot 2h}) + \dots + \{(n-1)h + e^{2(n-1)h}\}] \\ &= 4 \lim_{n \rightarrow \infty} \frac{1}{n} [1 + (h + e^{2h}) + (2h + e^{4h}) + \dots + \{(n-1)h + e^{2(n-1)h}\}] \\ &= 4 \lim_{n \rightarrow \infty} \frac{1}{n} [\{h + 2h + 3h + \dots + (n-1)h\} + \{1 + e^{2h} + e^{4h} + \dots + e^{2(n-1)h}\}] \\ &= 4 \lim_{n \rightarrow \infty} \frac{1}{n} \left[h \{1 + 2 + \dots + (n-1)\} + \left(\frac{e^{2hn} - 1}{e^{2h} - 1} \right) \right] \\ &= 4 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{(h(n-1)n)}{2} + \left(\frac{e^{2hn} - 1}{e^{2h} - 1} \right) \right] \\ &= 4 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{4}{n} \cdot \frac{(n-1)n}{2} + \left(\frac{e^8 - 1}{e^{\frac{8}{n}} - 1} \right) \right] \\ &= 4(2) + 4 \lim_{n \rightarrow \infty} \frac{(e^8 - 1)}{\left(\frac{\frac{8}{n} - 1}{\frac{8}{n}} \right)} \cdot 8 \\ &= 8 + \frac{4 \cdot (e^8 - 1)}{8} \qquad \left(\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \right) \\ &= 8 + \frac{e^8 - 1}{2} \\ &= \frac{15 + e^8}{2} \end{aligned}$$

Exercise 7.9

Question 1:

$$\int_{-1}^1 (x+1) dx$$

Answer

$$\text{Let } I = \int_{-1}^1 (x+1) dx$$

$$\int (x+1) dx = \frac{x^2}{2} + x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= F(1) - F(-1) \\ &= \left(\frac{1}{2} + 1\right) - \left(\frac{1}{2} - 1\right) \\ &= \frac{1}{2} + 1 - \frac{1}{2} + 1 \\ &= 2 \end{aligned}$$

Question 2:

$$\int_2^3 \frac{1}{x} dx$$

Answer

$$\text{Let } I = \int_2^3 \frac{1}{x} dx$$

$$\int \frac{1}{x} dx = \log|x| = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(3) - F(2)$$

$$= \log|3| - \log|2| = \log \frac{3}{2}$$

Question 3:

$$\int_1^2 (4x^3 - 5x^2 + 6x + 9) dx$$

Answer

$$\text{Let } I = \int_1^2 (4x^3 - 5x^2 + 6x + 9) dx$$

$$\begin{aligned} \int (4x^3 - 5x^2 + 6x + 9) dx &= 4 \left(\frac{x^4}{4} \right) - 5 \left(\frac{x^3}{3} \right) + 6 \left(\frac{x^2}{2} \right) + 9(x) \\ &= x^4 - \frac{5x^3}{3} + 3x^2 + 9x = F(x) \end{aligned}$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= F(2) - F(1) \\ I &= \left\{ 2^4 - \frac{5 \cdot (2)^3}{3} + 3(2)^2 + 9(2) \right\} - \left\{ (1)^4 - \frac{5(1)^3}{3} + 3(1)^2 + 9(1) \right\} \\ &= \left(16 - \frac{40}{3} + 12 + 18 \right) - \left(1 - \frac{5}{3} + 3 + 9 \right) \\ &= 16 - \frac{40}{3} + 12 + 18 - 1 + \frac{5}{3} - 3 - 9 \\ &= 33 - \frac{35}{3} \\ &= \frac{99 - 35}{3} \\ &= \frac{64}{3} \end{aligned}$$

Question 4:

$$\int_0^{\frac{\pi}{4}} \sin 2x dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \sin 2x dx$$

$$\int \sin 2x dx = \left(\frac{-\cos 2x}{2} \right) = F(x)$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= F\left(\frac{\pi}{4}\right) - F(0) \\ &= -\frac{1}{2} \left[\cos 2\left(\frac{\pi}{4}\right) - \cos 0 \right] \\ &= -\frac{1}{2} \left[\cos\left(\frac{\pi}{2}\right) - \cos 0 \right] \\ &= -\frac{1}{2} [0 - 1] \\ &= \frac{1}{2} \end{aligned}$$

Question 5:

$$\int_0^{\frac{\pi}{2}} \cos 2x \, dx$$

Answer

$$\begin{aligned} \text{Let } I &= \int_0^{\frac{\pi}{2}} \cos 2x \, dx \\ \int \cos 2x \, dx &= \left(\frac{\sin 2x}{2} \right) = F(x) \end{aligned}$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= F\left(\frac{\pi}{2}\right) - F(0) \\ &= \frac{1}{2} \left[\sin 2\left(\frac{\pi}{2}\right) - \sin 0 \right] \\ &= \frac{1}{2} [\sin \pi - \sin 0] \\ &= \frac{1}{2} [0 - 0] = 0 \end{aligned}$$

Question 6:

$$\int_4^5 e^x dx$$

Answer

$$\text{Let } I = \int_4^5 e^x dx$$

$$\int e^x dx = e^x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= F(5) - F(4) \\ &= e^5 - e^4 \\ &= e^4(e-1) \end{aligned}$$

Question 7:

$$\int_0^{\frac{\pi}{4}} \tan x dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \tan x dx$$

$$\int \tan x dx = -\log |\cos x| = F(x)$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= F\left(\frac{\pi}{4}\right) - F(0) \\ &= -\log \left| \cos \frac{\pi}{4} \right| + \log |\cos 0| \\ &= -\log \left| \frac{1}{\sqrt{2}} \right| + \log |1| \\ &= -\log (2)^{-\frac{1}{2}} \\ &= \frac{1}{2} \log 2 \end{aligned}$$

Question 8:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$$

Answer

$$\text{Let } I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$$

$$\int \operatorname{cosec} x \, dx = \log |\operatorname{cosec} x - \cot x| = F(x)$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= F\left(\frac{\pi}{4}\right) - F\left(\frac{\pi}{6}\right) \\ &= \log \left| \operatorname{cosec} \frac{\pi}{4} - \cot \frac{\pi}{4} \right| - \log \left| \operatorname{cosec} \frac{\pi}{6} - \cot \frac{\pi}{6} \right| \\ &= \log |\sqrt{2} - 1| - \log |2 - \sqrt{3}| \\ &= \log \left(\frac{\sqrt{2} - 1}{2 - \sqrt{3}} \right) \end{aligned}$$

Question 9:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

Answer

$$\text{Let } I = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= F(1) - F(0) \\ &= \sin^{-1}(1) - \sin^{-1}(0) \\ &= \frac{\pi}{2} - 0 \\ &= \frac{\pi}{2} \end{aligned}$$

Question 10:

$$\int_0^1 \frac{dx}{1+x^2}$$

Answer

$$\text{Let } I = \int_0^1 \frac{dx}{1+x^2}$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= F(1) - F(0) \\ &= \tan^{-1}(1) - \tan^{-1}(0) \\ &= \frac{\pi}{4} \end{aligned}$$

Question 11:

$$\int_2^3 \frac{dx}{x^2 - 1}$$

Answer

$$\text{Let } I = \int_2^3 \frac{dx}{x^2 - 1}$$

$$\int \frac{dx}{x^2 - 1} = \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| = F(x)$$

$$I = F(3) - F(2)$$

$$= \frac{1}{2} \left[\log \left| \frac{3-1}{3+1} \right| - \log \left| \frac{2-1}{2+1} \right| \right]$$

$$= \frac{1}{2} \left[\log \left| \frac{2}{4} \right| - \log \left| \frac{1}{3} \right| \right]$$

$$= \frac{1}{2} \left[\log \frac{1}{2} - \log \frac{1}{3} \right]$$

$$= \frac{1}{2} \left[\log \frac{3}{2} \right]$$

B

y second fundamental theorem of

calculus, we obtain Question 12:

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$$

$$\int \cos^2 x \, dx = \int \left(\frac{1 + \cos 2x}{2} \right) dx = \frac{x}{2} + \frac{\sin 2x}{4} = \frac{1}{2} \left(x + \frac{\sin 2x}{2} \right) = F(x)$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= \left[F\left(\frac{\pi}{2}\right) - F(0) \right] \\ &= \frac{1}{2} \left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2} \right) - \left(0 + \frac{\sin 0}{2} \right) \right] \\ &= \frac{1}{2} \left[\frac{\pi}{2} + 0 - 0 - 0 \right] \\ &= \frac{\pi}{4} \end{aligned}$$

Question 13:

$$\int_2^3 \frac{x dx}{x^2 + 1}$$

Answer

$$\text{Let } I = \int_2^3 \frac{x}{x^2 + 1} dx$$

$$\int \frac{x}{x^2 + 1} dx = \frac{1}{2} \int \frac{2x}{x^2 + 1} dx = \frac{1}{2} \log(1 + x^2) = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(3) - F(2)$$

$$= \frac{1}{2} \left[\log(1 + (3)^2) - \log(1 + (2)^2) \right]$$

$$= \frac{1}{2} \left[\log(10) - \log(5) \right]$$

$$= \frac{1}{2} \log\left(\frac{10}{5}\right) = \frac{1}{2} \log 2$$

Question 14:

$$\int_0^1 \frac{2x+3}{5x^2+1} dx$$

Answer

$$\text{Let } I = \int_0^1 \frac{2x+3}{5x^2+1} dx$$

$$\begin{aligned} \int \frac{2x+3}{5x^2+1} dx &= \frac{1}{5} \int \frac{5(2x+3)}{5x^2+1} dx \\ &= \frac{1}{5} \int \frac{10x+15}{5x^2+1} dx \\ &= \frac{1}{5} \int \frac{10x}{5x^2+1} dx + 3 \int \frac{1}{5x^2+1} dx \\ &= \frac{1}{5} \int \frac{10x}{5x^2+1} dx + 3 \int \frac{1}{5\left(x^2+\frac{1}{5}\right)} dx \\ &= \frac{1}{5} \log(5x^2+1) + \frac{3}{5} \cdot \frac{1}{\sqrt{5}} \tan^{-1} \frac{x}{\frac{1}{\sqrt{5}}} \\ &= \frac{1}{5} \log(5x^2+1) + \frac{3}{\sqrt{5}} \tan^{-1}(\sqrt{5}x) \\ &= F(x) \end{aligned}$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= F(1) - F(0) \\ &= \left\{ \frac{1}{5} \log(5+1) + \frac{3}{\sqrt{5}} \tan^{-1}(\sqrt{5}) \right\} - \left\{ \frac{1}{5} \log(1) + \frac{3}{\sqrt{5}} \tan^{-1}(0) \right\} \\ &= \frac{1}{5} \log 6 + \frac{3}{\sqrt{5}} \tan^{-1} \sqrt{5} \end{aligned}$$

Question 15:

$$\int_0^1 xe^{x^2} dx$$

Answer

$$\text{Let } I = \int_0^1 x e^{x^2} dx$$

$$\text{Put } x^2 = t \Rightarrow 2x dx = dt$$

As $x \rightarrow 0, t \rightarrow 0$ and as $x \rightarrow 1, t \rightarrow 1$,

$$\therefore I = \frac{1}{2} \int_0^1 e^t dt$$

$$\frac{1}{2} \int e^t dt = \frac{1}{2} e^t = F(t)$$

By second fundamental theorem of calculus, we obtain

$$I = F(1) - F(0)$$

$$= \frac{1}{2} e - \frac{1}{2} e^0$$

$$= \frac{1}{2} (e - 1)$$

Question 16:

$$\int_0^1 \frac{5x^2}{x^2 + 4x + 3}$$

Answer

$$I = \int_1^2 \frac{5x^2}{x^2 + 4x + 3} dx$$

Let

Dividing $5x^2$ by $x^2 + 4x + 3$, we obtain

$$\begin{aligned} I &= \int_1^2 \left\{ 5 - \frac{20x + 15}{x^2 + 4x + 3} \right\} dx \\ &= \int_1^2 5 dx - \int_1^2 \frac{20x + 15}{x^2 + 4x + 3} dx \\ &= [5x]_1^2 - \int_1^2 \frac{20x + 15}{x^2 + 4x + 3} dx \end{aligned}$$

$$I = 5 - I_1, \text{ where } I = \int_1^2 \frac{20x + 15}{x^2 + 4x + 3} dx \quad \dots(1)$$

$$\text{Consider } I_1 = \int_1^2 \frac{20x + 15}{x^2 + 4x + 3} dx$$

$$\begin{aligned} \text{Let } 20x + 15 &= A \frac{d}{dx} (x^2 + 4x + 3) + B \\ &= 2Ax + (4A + B) \end{aligned}$$

Equating the coefficients of x and constant term, we obtain

$$A = 10 \text{ and } B = -25$$

$$\Rightarrow I_1 = 10 \int_1^2 \frac{2x+4}{x^2+4x+3} dx - 25 \int_1^2 \frac{dx}{x^2+4x+3}$$

$$\text{Let } x^2 + 4x + 3 = t$$

$$\Rightarrow (2x+4) dx = dt$$

$$\begin{aligned} \Rightarrow I_1 &= 10 \int \frac{dt}{t} - 25 \int \frac{dx}{(x+2)^2 - 1^2} \\ &= 10 \log t - 25 \left[\frac{1}{2} \log \left(\frac{x+2-1}{x+2+1} \right) \right] \\ &= \left[10 \log (x^2 + 4x + 3) \right]_1^2 - 25 \left[\frac{1}{2} \log \left(\frac{x+1}{x+3} \right) \right]_1^2 \\ &= [10 \log 15 - 10 \log 8] - 25 \left[\frac{1}{2} \log \frac{3}{5} - \frac{1}{2} \log \frac{2}{4} \right] \\ &= [10 \log (5 \times 3) - 10 \log (4 \times 2)] - \frac{25}{2} [\log 3 - \log 5 - \log 2 + \log 4] \\ &= [10 \log 5 + 10 \log 3 - 10 \log 4 - 10 \log 2] - \frac{25}{2} [\log 3 - \log 5 - \log 2 + \log 4] \\ &= \left[10 + \frac{25}{2} \right] \log 5 + \left[-10 - \frac{25}{2} \right] \log 4 + \left[10 - \frac{25}{2} \right] \log 3 + \left[-10 + \frac{25}{2} \right] \log 2 \\ &= \frac{45}{2} \log 5 - \frac{45}{2} \log 4 - \frac{5}{2} \log 3 + \frac{5}{2} \log 2 \\ &= \frac{45}{2} \log \frac{5}{4} - \frac{5}{2} \log \frac{3}{2} \end{aligned}$$

Substituting the value of I_1 in (1), we obtain

$$\begin{aligned} I &= 5 - \left[\frac{45}{2} \log \frac{5}{4} - \frac{5}{2} \log \frac{3}{2} \right] \\ &= 5 - \frac{5}{2} \left[9 \log \frac{5}{4} - \log \frac{3}{2} \right] \end{aligned}$$

Question 17:

$$\int_0^{\frac{\pi}{4}} (2 \sec^2 x + x^3 + 2) dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} (2 \sec^2 x + x^3 + 2) dx$$

$$\int (2 \sec^2 x + x^3 + 2) dx = 2 \tan x + \frac{x^4}{4} + 2x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= F\left(\frac{\pi}{4}\right) - F(0) \\ &= \left\{ \left(2 \tan \frac{\pi}{4} + \frac{1}{4} \left(\frac{\pi}{4} \right)^4 + 2 \left(\frac{\pi}{4} \right) \right) - (2 \tan 0 + 0 + 0) \right\} \\ &= 2 \tan \frac{\pi}{4} + \frac{\pi^4}{4^5} + \frac{\pi}{2} \\ &= 2 + \frac{\pi}{2} + \frac{\pi^4}{1024} \end{aligned}$$

Question 18:

$$\int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx$$

Answer

$$\begin{aligned}\text{Let } I &= \int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx \\ &= - \int_0^{\pi} \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right) dx \\ &= - \int_0^{\pi} \cos x \, dx \\ \int \cos x \, dx &= \sin x = F(x)\end{aligned}$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned}I &= F(\pi) - F(0) \\ &= \sin \pi - \sin 0 \\ &= 0\end{aligned}$$

Question 19:

$$\int_0^2 \frac{6x+3}{x^2+4} dx$$

Answer

$$\begin{aligned}\text{Let } I &= \int_0^2 \frac{6x+3}{x^2+4} dx \\ \int \frac{6x+3}{x^2+4} dx &= 3 \int \frac{2x+1}{x^2+4} dx \\ &= 3 \int \frac{2x}{x^2+4} dx + 3 \int \frac{1}{x^2+4} dx \\ &= 3 \log(x^2+4) + \frac{3}{2} \tan^{-1} \frac{x}{2} = F(x)\end{aligned}$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned}
 I &= F(2) - F(0) \\
 &= \left\{ 3 \log(2^2 + 4) + \frac{3}{2} \tan^{-1} \left(\frac{2}{2} \right) \right\} - \left\{ 3 \log(0 + 4) + \frac{3}{2} \tan^{-1} \left(\frac{0}{2} \right) \right\} \\
 &= 3 \log 8 + \frac{3}{2} \tan^{-1} 1 - 3 \log 4 - \frac{3}{2} \tan^{-1} 0 \\
 &= 3 \log 8 + \frac{3}{2} \left(\frac{\pi}{4} \right) - 3 \log 4 - 0 \\
 &= 3 \log \left(\frac{8}{4} \right) + \frac{3\pi}{8} \\
 &= 3 \log 2 + \frac{3\pi}{8}
 \end{aligned}$$

Question 20:

$$\int_0^1 \left(xe^x + \sin \frac{\pi x}{4} \right) dx$$

Answer

$$\text{Let } I = \int_0^1 \left(xe^x + \sin \frac{\pi x}{4} \right) dx$$

$$\begin{aligned}
 \int \left(xe^x + \sin \frac{\pi x}{4} \right) dx &= x \int e^x dx - \int \left\{ \left(\frac{d}{dx} x \right) \int e^x dx \right\} dx + \left\{ \frac{-\cos \frac{\pi x}{4}}{\frac{\pi}{4}} \right\} \\
 &= xe^x - \int e^x dx - \frac{4\pi}{\pi} \cos \frac{x}{4} \\
 &= xe^x - e^x - \frac{4\pi}{\pi} \cos \frac{x}{4} \\
 &= F(x)
 \end{aligned}$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= F(1) - F(0) \\ &= \left(1.e^1 - e^1 - \frac{4}{\pi} \cos \frac{\pi}{4} \right) - \left(0.e^0 - e^0 - \frac{4}{\pi} \cos 0 \right) \\ &= e - e - \frac{4}{\pi} \left(\frac{1}{\sqrt{2}} \right) + 1 + \frac{4}{\pi} \\ &= 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi} \end{aligned}$$

Question 21:

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} \text{ equals}$$

- A. $\frac{\pi}{3}$
- B. $\frac{2\pi}{3}$
- C. $\frac{\pi}{6}$
- D. $\frac{\pi}{12}$

Answer

$$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$$

$$\begin{aligned} \int_1^{\sqrt{3}} \frac{dx}{1+x^2} &= F(\sqrt{3}) - F(1) \\ &= \tan^{-1} \sqrt{3} - \tan^{-1} 1 \\ &= \frac{\pi}{3} - \frac{\pi}{4} \\ &= \frac{\pi}{12} \end{aligned}$$

Hence, the correct Answer is D.

By second fundamental theorem of calculus, we obtain

Question 22:

$$\int_1^2 \frac{dx}{4+9x^2} \text{ equals}$$

- A. $\frac{\pi}{6}$
- B. $\frac{\pi}{12}$
- C. $\frac{\pi}{24}$
- D. $\frac{\pi}{4}$

Answer

$$\int \frac{dx}{4+9x^2} = \int \frac{dx}{(2)^2 + (3x)^2}$$

$$\text{Put } 3x = t \Rightarrow 3dx = dt$$

$$\begin{aligned} \therefore \int \frac{dx}{(2)^2 + (3x)^2} &= \frac{1}{3} \int \frac{dt}{(2)^2 + t^2} \\ &= \frac{1}{3} \left[\frac{1}{2} \tan^{-1} \frac{t}{2} \right] \\ &= \frac{1}{6} \tan^{-1} \left(\frac{3x}{2} \right) \\ &= F(x) \end{aligned}$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned}\int_0^2 \frac{dx}{4+9x^2} &= F\left(\frac{2}{3}\right) - F(0) \\ &= \frac{1}{6} \tan^{-1}\left(\frac{3}{2} \cdot \frac{2}{3}\right) - \frac{1}{6} \tan^{-1} 0 \\ &= \frac{1}{6} \tan^{-1} 1 - 0 \\ &= \frac{1}{6} \times \frac{\pi}{4} \\ &= \frac{\pi}{24}\end{aligned}$$

Hence, the correct Answer is C.

Exercise 7.10

Question 1:

$$\int_0^1 \frac{x}{x^2+1} dx$$

Answer

$$\int_0^1 \frac{x}{x^2+1} dx$$

$$\text{Let } x^2 + 1 = t \Rightarrow 2x dx = dt$$

When $x = 0$, $t = 1$ and when $x = 1$, $t = 2$

$$\begin{aligned} \therefore \int_0^1 \frac{x}{x^2+1} dx &= \frac{1}{2} \int_1^2 \frac{dt}{t} \\ &= \frac{1}{2} [\log |t|]_1^2 \\ &= \frac{1}{2} [\log 2 - \log 1] \\ &= \frac{1}{2} \log 2 \end{aligned}$$

Question 2:

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin \phi} \cos^5 \phi d\phi$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \sqrt{\sin \phi} \cos^5 \phi d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin \phi} \cos^4 \phi \cos \phi d\phi$$

$$\text{Also, let } \sin \phi = t \Rightarrow \cos \phi d\phi = dt$$

When $\phi = 0, t = 0$ and when $\phi = \frac{\pi}{2}, t = 1$

$$\begin{aligned} \therefore I &= \int_0^1 \sqrt{t}(1-t^2)^2 dt \\ &= \int_0^1 t^{\frac{1}{2}}(1+t^4-2t^2) dt \\ &= \int_0^1 \left[t^{\frac{1}{2}} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}} \right] dt \\ &= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} - \frac{2t^{\frac{7}{2}}}{\frac{7}{2}} \right]_0^1 \\ &= \frac{2}{3} + \frac{2}{11} - \frac{4}{7} \\ &= \frac{154+42-132}{231} \\ &= \frac{64}{231} \end{aligned}$$

Question 3:

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$$

Answer

$$\text{Let } I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$$

Also, let $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

When $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{\pi}{4}$

$$\begin{aligned}
 I &= \int_0^{\frac{\pi}{4}} \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) \sec^2 \theta d\theta \\
 &= \int_0^{\frac{\pi}{4}} \sin^{-1} (\sin 2\theta) \sec^2 \theta d\theta \\
 &= \int_0^{\frac{\pi}{4}} 2\theta \cdot \sec^2 \theta d\theta \\
 &= 2 \int_0^{\frac{\pi}{4}} \theta \cdot \sec^2 \theta d\theta
 \end{aligned}$$

Taking θ as first function and $\sec^2 \theta$ as second function and integrating by parts, we obtain

$$\begin{aligned}
 I &= 2 \left[\theta \int \sec^2 \theta d\theta - \int \left\{ \left(\frac{d}{dx} \theta \right) \int \sec^2 \theta d\theta \right\} d\theta \right]_0^{\frac{\pi}{4}} \\
 &= 2 \left[\theta \tan \theta - \int \tan \theta d\theta \right]_0^{\frac{\pi}{4}} \\
 &= 2 \left[\theta \tan \theta + \log |\cos \theta| \right]_0^{\frac{\pi}{4}} \\
 &= 2 \left[\frac{\pi}{4} \tan \frac{\pi}{4} + \log \left| \cos \frac{\pi}{4} \right| - \log |\cos 0| \right] \\
 &= 2 \left[\frac{\pi}{4} + \log \left(\frac{1}{\sqrt{2}} \right) - \log 1 \right] \\
 &= 2 \left[\frac{\pi}{4} - \frac{1}{2} \log 2 \right] \\
 &= \frac{\pi}{2} - \log 2
 \end{aligned}$$

Question 4:

$$\int_0^2 x\sqrt{x+2} \quad (\text{Put } x+2=t^2)$$

Answer

$$\int_0^2 x\sqrt{x+2} dx$$

$$\text{Let } x+2=t^2 \Rightarrow dx=2t dt$$

When $x=0$, $t=\sqrt{2}$ and when $x=2$, $t=2$

$$\begin{aligned} \therefore \int_0^2 x\sqrt{x+2} dx &= \int_{\sqrt{2}}^2 (t^2-2)\sqrt{t^2} 2t dt \\ &= 2 \int_{\sqrt{2}}^2 (t^2-2)t^2 dt \\ &= 2 \int_{\sqrt{2}}^2 (t^4-2t^2) dt \\ &= 2 \left[\frac{t^5}{5} - \frac{2t^3}{3} \right]_{\sqrt{2}}^2 \\ &= 2 \left[\frac{32}{5} - \frac{16}{3} - \frac{4\sqrt{2}}{5} + \frac{4\sqrt{2}}{3} \right] \\ &= 2 \left[\frac{96-80-12\sqrt{2}+20\sqrt{2}}{15} \right] \\ &= 2 \left[\frac{16+8\sqrt{2}}{15} \right] \\ &= \frac{16(2+\sqrt{2})}{15} \\ &= \frac{16\sqrt{2}(\sqrt{2}+1)}{15} \end{aligned}$$

$$\begin{aligned} \Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x} dx &= - \int_1^0 \frac{dt}{1+t^2} \\ &= - \left[\tan^{-1} t \right]_1^0 \\ &= - \left[\tan^{-1} 0 - \tan^{-1} 1 \right] \\ &= - \left[-\frac{\pi}{4} \right] \\ &= \frac{\pi}{4} \end{aligned}$$

Question 5:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x} dx$$

Answer

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x} dx$$

Let $\cos x = t \Rightarrow -\sin x dx = dt$

When $x = 0$, $t = 1$ and when $x = \frac{\pi}{2}$, $t = 0$

Question 6:

$$\int_0^2 \frac{dx}{x+4-x^2}$$

Answer

$$\begin{aligned} \int_0^2 \frac{dx}{x+4-x^2} &= \int_0^2 \frac{dx}{-(x^2-x-4)} \\ &= \int_0^2 \frac{dx}{-\left(x^2-x+\frac{1}{4}-\frac{1}{4}-4\right)} \\ &= \int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2-\frac{17}{4}\right]} \\ &= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2} \end{aligned}$$

Let $x - \frac{1}{2} = t \Rightarrow dx = dt$

When $x = 0$, $t = -\frac{1}{2}$ and when $x = 2$, $t = \frac{3}{2}$

$$\begin{aligned} \therefore \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} &= \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2 - t^2} \\ &= \left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)} \log \frac{\frac{\sqrt{17}}{2} + t}{\frac{\sqrt{17}}{2} - t} \right]_{-\frac{1}{2}}^{\frac{3}{2}} \\ &= \frac{1}{\sqrt{17}} \left[\log \frac{\frac{\sqrt{17}}{2} + \frac{3}{2}}{\frac{\sqrt{17}}{2} - \frac{3}{2}} - \log \frac{\frac{\sqrt{17}}{2} - \frac{1}{2}}{\frac{\sqrt{17}}{2} + \frac{1}{2}} \right] \\ &= \frac{1}{\sqrt{17}} \left[\log \frac{\sqrt{17} + 3}{\sqrt{17} - 3} - \log \frac{\sqrt{17} - 1}{\sqrt{17} + 1} \right] \\ &= \frac{1}{\sqrt{17}} \log \frac{\sqrt{17} + 3}{\sqrt{17} - 3} \times \frac{\sqrt{17} + 1}{\sqrt{17} - 1} \\ &= \frac{1}{\sqrt{17}} \log \left[\frac{17 + 3 + 4\sqrt{17}}{17 + 3 - 4\sqrt{17}} \right] \\ &= \frac{1}{\sqrt{17}} \log \left[\frac{20 + 4\sqrt{17}}{20 - 4\sqrt{17}} \right] \\ &= \frac{1}{\sqrt{17}} \log \left(\frac{5 + \sqrt{17}}{5 - \sqrt{17}} \right) \\ &= \frac{1}{\sqrt{17}} \log \left[\frac{(5 + \sqrt{17})(5 + \sqrt{17})}{25 - 17} \right] \\ &= \frac{1}{\sqrt{17}} \log \left[\frac{25 + 17 + 10\sqrt{17}}{8} \right] \\ &= \frac{1}{\sqrt{17}} \log \left(\frac{42 + 10\sqrt{17}}{8} \right) \\ &= \frac{1}{\sqrt{17}} \log \left(\frac{21 + 5\sqrt{17}}{4} \right) \end{aligned}$$

Question 7:

$$\int_{-1}^1 \frac{dx}{x^2 + 2x + 5}$$

Answer

$$\int_{-1}^1 \frac{dx}{x^2 + 2x + 5} = \int_{-1}^1 \frac{dx}{(x^2 + 2x + 1) + 4} = \int_{-1}^1 \frac{dx}{(x+1)^2 + (2)^2}$$

Let $x + 1 = t \Rightarrow dx = dt$

When $x = -1$, $t = 0$ and when $x = 1$, $t = 2$

$$\begin{aligned} \therefore \int_{-1}^1 \frac{dx}{(x+1)^2 + (2)^2} &= \int_0^2 \frac{dt}{t^2 + 2^2} \\ &= \left[\frac{1}{2} \tan^{-1} \frac{t}{2} \right]_0^2 \\ &= \frac{1}{2} \tan^{-1} 1 - \frac{1}{2} \tan^{-1} 0 \\ &= \frac{1}{2} \left(\frac{\pi}{4} \right) = \frac{\pi}{8} \end{aligned}$$

Question 8:

$$\int \left(\frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx$$

Answer

$$\int \left(\frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx$$

Let $2x = t \Rightarrow 2dx = dt$

When $x = 1$, $t = 2$ and when $x = 2$, $t = 4$

$$\begin{aligned} \therefore \int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx &= \frac{1}{2} \int_2^4 \left(\frac{2}{t} - \frac{2}{t^2} \right) e^t dt \\ &= \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t dt \end{aligned}$$

Let $\frac{1}{t} = f(t)$

Then, $f'(t) = -\frac{1}{t^2}$

$$\begin{aligned} \Rightarrow \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t dt &= \int_2^4 e^t [f(t) + f'(t)] dt \\ &= [e^t f(t)]_2^4 \\ &= \left[e^t \cdot \frac{2}{t} \right]_2^4 \\ &= \left[\frac{e^t}{t} \right]_2^4 \\ &= \frac{e^4}{4} - \frac{e^2}{2} \\ &= \frac{e^2(e^2 - 2)}{4} \end{aligned}$$

Question 9:

The value of the integral $\int_3^4 \frac{(x-x^3)^{\frac{1}{3}}}{x^4} dx$ is

- A. 6
- B. 0
- C. 3
- D. 4

$$\text{Let } I = \int_{\frac{1}{3}}^1 \frac{(x-x^3)^{\frac{1}{3}}}{x^4} dx$$

$$\text{Also, let } x = \sin \theta \Rightarrow dx = \cos \theta d\theta$$

$$\text{When } x = \frac{1}{3}, \theta = \sin^{-1}\left(\frac{1}{3}\right) \text{ and when } x = 1, \theta = \frac{\pi}{2}$$

$$\begin{aligned} \Rightarrow I &= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin \theta - \sin^3 \theta)^{\frac{1}{3}}}{\sin^4 \theta} \cos \theta d\theta \\ &= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin \theta)^{\frac{1}{3}} (1 - \sin^2 \theta)^{\frac{1}{3}}}{\sin^4 \theta} \cos \theta d\theta \\ &= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin \theta)^{\frac{1}{3}} (\cos \theta)^{\frac{2}{3}}}{\sin^4 \theta} \cos \theta d\theta \\ &= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin \theta)^{\frac{1}{3}} (\cos \theta)^{\frac{2}{3}}}{\sin^2 \theta \sin^2 \theta} \cos \theta d\theta \\ &= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\cos \theta)^{\frac{5}{3}}}{(\sin \theta)^{\frac{5}{3}}} \operatorname{cosec}^2 \theta d\theta \\ &= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} (\cot \theta)^{\frac{5}{3}} \operatorname{cosec}^2 \theta d\theta \end{aligned}$$

Let $\cot \theta = t \Rightarrow -\operatorname{cosec}^2 \theta d\theta = dt$
 Answer

When $\theta = \sin^{-1}\left(\frac{1}{3}\right)$, $t = 2\sqrt{2}$ and when $\theta = \frac{\pi}{2}$, $t = 0$

$$\begin{aligned} \therefore I &= -\int_{2\sqrt{2}}^0 (t)^{\frac{5}{3}} dt \\ &= -\left[\frac{3}{8}(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^0 \\ &= -\frac{3}{8}\left[(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^0 \\ &= -\frac{3}{8}\left[-(2\sqrt{2})^{\frac{8}{3}}\right] \\ &= \frac{3}{8}\left[(\sqrt{8})^{\frac{8}{3}}\right] \\ &= \frac{3}{8}\left[(8)^{\frac{4}{3}}\right] \\ &= \frac{3}{8}[16] \\ &= 3 \times 2 \\ &= 6 \end{aligned}$$

Hence, the correct Answer is A.

Question 10:

If $f(x) = \int_0^x t \sin t dt$, then $f'(x)$ is

- A. $\cos x + x \sin x$
- B. $x \sin x$
- C. $x \cos x$
- D. $\sin x + x \cos x$

Answer

$$f(x) = \int_0^x t \sin t dt$$

Integrating by parts, we obtain

$$\begin{aligned} f(x) &= t \int_0^x \sin t \, dt - \int_0^x \left\{ \left(\frac{d}{dt} t \right) \int \sin t \, dt \right\} dt \\ &= [t(-\cos t)]_0^x - \int_0^x (-\cos t) dt \\ &= [-t \cos t + \sin t]_0^x \\ &= -x \cos x + \sin x \end{aligned}$$

$$\begin{aligned} \Rightarrow f'(x) &= -[x(-\sin x) + \cos x] + \cos x \\ &= x \sin x - \cos x + \cos x \\ &= x \sin x \end{aligned}$$

Hence, the correct Answer is B.

Exercise 7.11

Question 1:

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$$

Answer

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx \quad \dots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \cos^2 \left(\frac{\pi}{2} - x \right) dx \quad \left(\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx \right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2 x \, dx \quad \dots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x) \, dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \, dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 2:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, dx$$

Answer

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \dots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \dots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 3:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^2 x + \cos^2 x} dx \quad \dots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)}{\sin^2\left(\frac{\pi}{2} - x\right) + \cos^2\left(\frac{\pi}{2} - x\right)} dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{3}{2}} x}{\sin^2 x + \cos^2 x} dx \quad \dots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sin^2 x + \cos^2 x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 4:

$$\int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x}$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} dx \quad \dots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 \left(\frac{\pi}{2} - x \right)}{\sin^5 \left(\frac{\pi}{2} - x \right) + \cos^5 \left(\frac{\pi}{2} - x \right)} dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx \quad \dots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 5:

$$\int_{-5}^5 |x+2| dx$$

Answer

$$\text{Let } I = \int_{-5}^5 |x+2| dx$$

It can be seen that $(x + 2) \leq 0$ on $[-5, -2]$ and $(x + 2) \geq 0$ on $[-2, 5]$.

$$\begin{aligned} \therefore I &= \int_{-5}^{-2} -(x+2)dx + \int_{-2}^5 (x+2)dx && \left(\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x) \right) \\ I &= -\left[\frac{x^2}{2} + 2x \right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x \right]_{-2}^5 \\ &= -\left[\frac{(-2)^2}{2} + 2(-2) - \frac{(-5)^2}{2} - 2(-5) \right] + \left[\frac{(5)^2}{2} + 2(5) - \frac{(-2)^2}{2} - 2(-2) \right] \\ &= -\left[2 - 4 - \frac{25}{2} + 10 \right] + \left[\frac{25}{2} + 10 - 2 + 4 \right] \\ &= -2 + 4 + \frac{25}{2} - 10 + \frac{25}{2} + 10 - 2 + 4 \\ &= 29 \end{aligned}$$

Question 6:

$$\int_2^8 |x-5| dx$$

Answer

$$\text{Let } I = \int_2^8 |x-5| dx$$

It can be seen that $(x - 5) \leq 0$ on $[2, 5]$ and $(x - 5) \geq 0$ on $[5, 8]$.

$$\begin{aligned} I &= \int_2^5 -(x-5)dx + \int_5^8 (x-5)dx && \left(\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x) \right) \\ &= -\left[\frac{x^2}{2} - 5x \right]_2^5 + \left[\frac{x^2}{2} - 5x \right]_5^8 \\ &= -\left[\frac{25}{2} - 25 - 2 + 10 \right] + \left[32 - 40 - \frac{25}{2} + 25 \right] \\ &= 9 \end{aligned}$$

Question 7:

$$\int_0^1 x(1-x)^n dx$$

Answer

$$\text{Let } I = \int_0^1 x(1-x)^n dx$$

$$\therefore I = \int_0^1 (1-x)(1-(1-x))^n dx$$

$$= \int_0^1 (1-x)(x)^n dx$$

$$= \int_0^1 (x^n - x^{n+1}) dx$$

$$= \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2} \right]_0^1$$

$$= \left[\frac{1}{n+1} - \frac{1}{n+2} \right]$$

$$= \frac{(n+2) - (n+1)}{(n+1)(n+2)}$$

$$= \frac{1}{(n+1)(n+2)}$$

$$\left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

Question 8:

$$\int_0^{\frac{\pi}{4}} \log (1 + \tan x) dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \log (1 + \tan x) dx \quad \dots(1)$$

$$\therefore I = \int_0^{\frac{\pi}{4}} \log \left[1 + \tan \left(\frac{\pi}{4} - x \right) \right] dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log \left\{ 1 + \frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \tan x} \right\} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log \left\{ 1 + \frac{1 - \tan x}{1 + \tan x} \right\} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log \frac{2}{(1 + \tan x)} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log 2 dx - \int_0^{\frac{\pi}{4}} \log (1 + \tan x) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log 2 dx - I \quad [\text{From (1)}]$$

$$\Rightarrow 2I = [x \log 2]_0^{\frac{\pi}{4}}$$

$$\Rightarrow 2I = \frac{\pi}{4} \log 2$$

$$\Rightarrow I = \frac{\pi}{8} \log 2$$

Question 9:

$$\int_0^2 x\sqrt{2-x} dx$$

Answer

$$\text{Let } I = \int_0^2 x\sqrt{2-x} dx$$

$$I = \int_0^2 (2-x)\sqrt{x} dx$$

$$= \int_0^2 \left(2x^{\frac{1}{2}} - x^{\frac{3}{2}} \right) dx$$

$$= \left[2 \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right) - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} \right]_0^2$$

$$= \left[\frac{4}{3} x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}} \right]_0^2$$

$$= \frac{4}{3} (2)^{\frac{3}{2}} - \frac{2}{5} (2)^{\frac{5}{2}}$$

$$= \frac{4 \times 2\sqrt{2}}{3} - \frac{2}{5} \times 4\sqrt{2}$$

$$= \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}$$

$$= \frac{40\sqrt{2} - 24\sqrt{2}}{15}$$

$$= \frac{16\sqrt{2}}{15}$$

$$\left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

Question 10:

$$\int_0^{\frac{\pi}{2}} (2 \log \sin x - \log \sin 2x) dx$$

Answer

$$\begin{aligned} \text{Let } I &= \int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin 2x) dx \\ \Rightarrow I &= \int_0^{\frac{\pi}{2}} \{2\log \sin x - \log(2 \sin x \cos x)\} dx \\ \Rightarrow I &= \int_0^{\frac{\pi}{2}} \{2\log \sin x - \log \sin x - \log \cos x - \log 2\} dx \\ \Rightarrow I &= \int_0^{\frac{\pi}{2}} \{\log \sin x - \log \cos x - \log 2\} dx \quad \dots(1) \end{aligned}$$

It is known that, $\left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{\log \cos x - \log \sin x - \log 2\} dx \quad \dots(2)$$

Adding (1) and (2), we obtain

$$\begin{aligned} 2I &= \int_0^{\frac{\pi}{2}} (-\log 2 - \log 2) dx \\ \Rightarrow 2I &= -2 \log 2 \int_0^{\frac{\pi}{2}} 1 dx \\ \Rightarrow I &= -\log 2 \left[\frac{\pi}{2} \right] \\ \Rightarrow I &= \frac{\pi}{2} (-\log 2) \\ \Rightarrow I &= \frac{\pi}{2} \left[\log \frac{1}{2} \right] \\ \Rightarrow I &= \frac{\pi}{2} \log \frac{1}{2} \end{aligned}$$

Question 11:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \, dx$$

Answer

$$\text{Let } I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \, dx$$

As $\sin^2(-x) = (\sin(-x))^2 = (-\sin x)^2 = \sin^2 x$, therefore, $\sin^2 x$ is an even function.

It is known that if $f(x)$ is an even function, then $\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx$

$$\begin{aligned} I &= 2 \int_0^{\frac{\pi}{2}} \sin^2 x \, dx \\ &= 2 \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2x}{2} \, dx \\ &= \int_0^{\frac{\pi}{2}} (1 - \cos 2x) \, dx \\ &= \left[x - \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} \end{aligned}$$

Question 12:

$$\int_0^{\pi} \frac{x \, dx}{1 + \sin x}$$

Answer

$$\text{Let } I = \int_0^{\pi} \frac{x \, dx}{1 + \sin x} \quad \dots(1)$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1 + \sin(\pi - x)} \, dx \quad \left(\int_0^a f(x) \, dx = \int_0^a f(a - x) \, dx \right)$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1 + \sin x} \, dx \quad \dots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\pi} \frac{\pi}{1 + \sin x} \, dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} \, dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x} \, dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \{ \sec^2 x - \tan x \sec x \} \, dx$$

$$\Rightarrow 2I = \pi [\tan x - \sec x]_0^{\pi}$$

$$\Rightarrow 2I = \pi [2]$$

$$\Rightarrow I = \pi$$

Question 13:

$$\int_{\frac{\pi}{2}}^{\pi} \sin^7 x \, dx$$

Answer

$$\text{Let } I = \int_{\frac{\pi}{2}}^{\pi} \sin^7 x \, dx \quad \dots(1)$$

As $\sin^7(-x) = (\sin(-x))^7 = (-\sin x)^7 = -\sin^7 x$, therefore, $\sin^7 x$ is an odd function.

It is known that, if $f(x)$ is an odd function, then $\int_{-a}^a f(x) dx = 0$

$$\therefore I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx = 0$$

Question 14:

$$\int_0^{2\pi} \cos^5 x dx$$

Answer

$$\text{Let } I = \int_0^{2\pi} \cos^5 x dx \quad \dots(1)$$

$$\cos^5(2\pi - x) = \cos^5 x$$

It is known that.

$$\begin{aligned} \int_0^{2a} f(x) dx &= 2 \int_0^a f(x) dx, \text{ if } f(2a-x) = f(x) \\ &= 0 \text{ if } f(2a-x) = -f(x) \end{aligned}$$

$$\therefore I = 2 \int_0^{\pi} \cos^5 x dx$$

$$\Rightarrow I = 2(0) = 0 \quad \left[\cos^5(\pi - x) = -\cos^5 x \right]$$

Question 15:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx \quad \dots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)} dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx \quad \dots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \frac{0}{1 + \sin x \cos x} dx$$

$$\Rightarrow I = 0$$

Question 16:

$$\int_0^{\pi} \log(1 + \cos x) dx$$

Answer

$$\text{Let } I = \int_0^{\pi} \log(1 + \cos x) dx \quad \dots(1)$$

$$\Rightarrow I = \int_0^{\pi} \log(1 + \cos(\pi - x)) dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx\right)$$

$$\Rightarrow I = \int_0^{\pi} \log(1 - \cos x) dx \quad \dots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\pi} \{ \log(1 + \cos x) + \log(1 - \cos x) \} dx$$

$$\Rightarrow 2I = \int_0^{\pi} \log(1 - \cos^2 x) dx$$

$$\Rightarrow 2I = \int_0^{\pi} \log \sin^2 x dx$$

$$\Rightarrow 2I = 2 \int_0^{\pi} \log \sin x dx$$

$$\Rightarrow I = \int_0^{\pi} \log \sin x dx \quad \dots(3)$$

$$\sin(\pi - x) = \sin x$$

$$\therefore I = 2 \int_0^{\frac{\pi}{2}} \log \sin x dx \quad \dots(4)$$

$$\Rightarrow I = 2 \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = 2 \int_0^{\frac{\pi}{2}} \log \cos x dx \quad \dots(5)$$

Adding (4) and (5), we obtain

$$2I = 2 \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x + \log 2 - \log 2) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log 2 \sin x \cos x - \log 2) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log \sin 2x dx - \int_0^{\frac{\pi}{2}} \log 2 dx$$

Let $2x = t \Rightarrow 2dx = dt$

When $x = 0$, $t = 0$ and when $x = \frac{\pi}{2}$, $t = \pi$

$$\therefore I = \frac{1}{2} \int_0^{\pi} \log \sin t dt - \frac{\pi}{2} \log 2$$

$$\Rightarrow I = \frac{1}{2} \int_0^{\pi} \log \sin t dt - \frac{\pi}{2} \log 2$$

$$\Rightarrow \frac{I}{2} = -\frac{\pi}{2} \log 2$$

$$\Rightarrow I = -\pi \log 2$$

Question 17:

$$\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$$

Answer

$$\text{Let } I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx \quad \dots(1)$$

It is known that, $\left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx \quad \dots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^a \frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}} dx$$

$$\Rightarrow 2I = \int_0^a 1 dx$$

$$\Rightarrow 2I = [x]_0^a$$

$$\Rightarrow 2I = a$$

$$\Rightarrow I = \frac{a}{2}$$

Question 18:

$$\int_0^4 |x-1| dx$$

Answer

$$I = \int_0^4 |x-1| dx$$

It can be seen that, $(x - 1) \leq 0$ when $0 \leq x \leq 1$ and $(x - 1) \geq 0$ when $1 \leq x \leq 4$

$$\begin{aligned}
 I &= \int_0^1 |x-1| dx + \int_1^4 |x-1| dx && \left(\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx \right) \\
 &= \int_0^1 -(x-1) dx + \int_1^4 (x-1) dx \\
 &= \left[x - \frac{x^2}{2} \right]_0^1 + \left[\frac{x^2}{2} - x \right]_1^4 \\
 &= 1 - \frac{1}{2} + \frac{(4)^2}{2} - 4 - \frac{1}{2} + 1 \\
 &= 1 - \frac{1}{2} + 8 - 4 - \frac{1}{2} + 1 \\
 &= 5
 \end{aligned}$$

Question 19:

Show that $\int_0^a f(x)g(x) dx = 2 \int_0^a f(x) dx$, if f and g are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$

Answer

$$\begin{aligned}
 \text{Let } I &= \int_0^a f(x)g(x) dx && \dots(1) \\
 \Rightarrow I &= \int_0^a f(a-x)g(a-x) dx && \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right) \\
 \Rightarrow I &= \int_0^a f(x)g(a-x) dx && \dots(2)
 \end{aligned}$$

Adding (1) and (2), we obtain

$$\begin{aligned}
 2I &= \int_0^a \{f(x)g(x) + f(x)g(a-x)\} dx \\
 \Rightarrow 2I &= \int_0^a f(x)\{g(x) + g(a-x)\} dx \\
 \Rightarrow 2I &= \int_0^a f(x) \times 4 dx && [g(x) + g(a-x) = 4] \\
 \Rightarrow I &= 2 \int_0^a f(x) dx
 \end{aligned}$$

Question 20:

The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x \cos x + \tan^5 x + 1) dx$ is

- A. 0
- B. 2
- C. π
- D. 1

Answer

$$\text{Let } I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x \cos x + \tan^5 x + 1) dx$$

$$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^5 x dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \cdot dx$$

It is known that if $f(x)$ is an even function, then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$ and

if $f(x)$ is an odd function, then $\int_{-a}^a f(x) dx = 0$

$$\begin{aligned} I &= 0 + 0 + 0 + 2 \int_0^{\frac{\pi}{2}} 1 \cdot dx \\ &= 2 \left[x \right]_0^{\frac{\pi}{2}} \\ &= \frac{2\pi}{2} \\ &= \pi \end{aligned}$$

Hence, the correct Answer is C.

Question 21:

The value of $\int_0^{\frac{\pi}{2}} \log \left(\frac{4 + 3 \sin x}{4 + 3 \cos x} \right) dx$ is

- A. 2
- B.

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \log \left(\frac{4 + 3 \sin x}{4 + 3 \cos x} \right) dx \quad \dots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log \left[\frac{4 + 3 \sin \left(\frac{\pi}{2} - x \right)}{4 + 3 \cos \left(\frac{\pi}{2} - x \right)} \right] dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log \left(\frac{4 + 3 \cos x}{4 + 3 \sin x} \right) dx \quad \dots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \left\{ \log \left(\frac{4 + 3 \sin x}{4 + 3 \cos x} \right) + \log \left(\frac{4 + 3 \cos x}{4 + 3 \sin x} \right) \right\} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log \left(\frac{4 + 3 \sin x}{4 + 3 \cos x} \times \frac{4 + 3 \cos x}{4 + 3 \sin x} \right) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log 1 dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0 dx$$

$$\Rightarrow I = 0$$

Hence, the correct Answer is C.

C. 0

D.

Question 1:

$$\frac{1}{x-x^3}$$

Answer

$$\frac{1}{x-x^3} = \frac{1}{x(1-x^2)} = \frac{1}{x(1-x)(1+x)}$$

$$\text{Let } \frac{1}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x} \quad \dots(1)$$

$$\Rightarrow 1 = A(1-x^2) + Bx(1+x) + Cx(1-x)$$

$$\Rightarrow 1 = A - Ax^2 + Bx + Bx^2 + Cx - Cx^2$$

Equating the coefficients of x^2 , x , and constant term, we obtain

$$-A + B - C = 0$$

$$B + C = 0$$

$$A = 1$$

On solving these equations, we obtain

$$A = 1, B = \frac{1}{2}, \text{ and } C = -\frac{1}{2}$$

From equation (1), we obtain

$$\begin{aligned}\frac{1}{x(1-x)(1+x)} &= \frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)} \\ \Rightarrow \int \frac{1}{x(1-x)(1+x)} dx &= \int \frac{1}{x} dx + \frac{1}{2} \int \frac{1}{1-x} dx - \frac{1}{2} \int \frac{1}{1+x} dx \\ &= \log|x| - \frac{1}{2} \log|(1-x)| - \frac{1}{2} \log|(1+x)| \\ &= \log|x| - \log\left|(1-x)^{\frac{1}{2}}\right| - \log\left|(1+x)^{\frac{1}{2}}\right| \\ &= \log\left|\frac{x}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}}\right| + C \\ &= \log\left|\left(\frac{x^2}{1-x^2}\right)^{\frac{1}{2}}\right| + C \\ &= \frac{1}{2} \log\left|\frac{x^2}{1-x^2}\right| + C\end{aligned}$$

Question 2:

$$\frac{1}{\sqrt{x+a} + \sqrt{(x+b)}}$$

Answer

$$\begin{aligned} \frac{1}{\sqrt{x+a}+\sqrt{x+b}} &= \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}} \\ &= \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)} \\ &= \frac{(\sqrt{x+a}-\sqrt{x+b})}{a-b} \end{aligned}$$

$$\begin{aligned} \Rightarrow \int \frac{1}{\sqrt{x+a}-\sqrt{x+b}} dx &= \frac{1}{a-b} \int (\sqrt{x+a}-\sqrt{x+b}) dx \\ &= \frac{1}{(a-b)} \left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}} \right] \\ &= \frac{2}{3(a-b)} \left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}} \right] + C \end{aligned}$$

Question 3:

$$\frac{1}{x\sqrt{ax-x^2}} \quad [\text{Hint: Put } x = \frac{a}{t}]$$

Answer

$$\frac{1}{x\sqrt{ax-x^2}}$$

$$\text{Let } x = \frac{a}{t} \Rightarrow dx = -\frac{a}{t^2} dt$$

$$\Rightarrow \int \frac{1}{x\sqrt{ax-x^2}} dx = \int \frac{1}{\frac{a}{t} \sqrt{a \cdot \frac{a}{t} - \left(\frac{a}{t}\right)^2}} \left(-\frac{a}{t^2} dt\right)$$

$$= -\int \frac{1}{at} \cdot \frac{1}{\sqrt{\frac{1}{t} - \frac{1}{t^2}}} dt$$

$$= -\frac{1}{a} \int \frac{1}{\sqrt{\frac{t^2}{t} - \frac{t^2}{t^2}}} dt$$

$$= -\frac{1}{a} \int \frac{1}{\sqrt{t-1}} dt$$

$$= -\frac{1}{a} [2\sqrt{t-1}] + C$$

$$= -\frac{1}{a} \left[2\sqrt{\frac{a}{x}-1} \right] + C$$

$$= -\frac{2}{a} \left(\frac{\sqrt{a-x}}{\sqrt{x}} \right) + C$$

$$= -\frac{2}{a} \left(\sqrt{\frac{a-x}{x}} \right) + C$$

Question 4:

$$\frac{1}{x^2(x^4+1)^{\frac{3}{4}}}$$

Answer

$$\frac{1}{x^2(x^4+1)^{\frac{3}{4}}}$$

Multiplying and dividing by x^{-3} , we obtain

$$\frac{x^{-3}}{x^2 \cdot x^{-3}(x^4+1)^{\frac{3}{4}}} = \frac{x^{-3}(x^4+1)^{-\frac{3}{4}}}{x^2 \cdot x^{-3}}$$

$$= \frac{(x^4+1)^{-\frac{3}{4}}}{x^5 \cdot (x^4)^{-\frac{3}{4}}}$$

$$= \frac{1}{x^5} \left(\frac{x^4+1}{x^4} \right)^{\frac{3}{4}}$$

$$= \frac{1}{x^5} \left(1 + \frac{1}{x^4} \right)^{\frac{3}{4}}$$

$$\text{Let } \frac{1}{x^4} = t \Rightarrow -\frac{4}{x^5} dx = dt \Rightarrow \frac{1}{x^5} dx = -\frac{dt}{4}$$

$$\therefore \int \frac{1}{x^2(x^4+1)^{\frac{3}{4}}} dx = \int \frac{1}{x^5} \left(1 + \frac{1}{x^4} \right)^{\frac{3}{4}} dx$$

$$= -\frac{1}{4} \int (1+t)^{\frac{3}{4}} dt$$

$$= -\frac{1}{4} \left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}} \right] + C$$

$$= -\frac{1}{4} \frac{\left(1 + \frac{1}{x^4} \right)^{\frac{1}{4}}}{\frac{1}{4}} + C$$

$$= -\left(1 + \frac{1}{x^4} \right)^{\frac{1}{4}} + C$$

Question 5:

$$\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} \left[\text{Hint: } \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}} \left(1 + x^{\frac{1}{6}} \right)} \text{ Put } x = t^6 \right]$$

Answer

$$\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}} \left(1 + x^{\frac{1}{6}} \right)}$$

$$\text{Let } x = t^6 \Rightarrow dx = 6t^5 dt$$

$$\begin{aligned} \therefore \int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} dx &= \int \frac{1}{x^{\frac{1}{3}} \left(1 + x^{\frac{1}{6}} \right)} dx \\ &= \int \frac{6t^5}{t^2(1+t)} dt \\ &= 6 \int \frac{t^3}{(1+t)} dt \end{aligned}$$

On dividing, we obtain

$$\begin{aligned} \int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} dx &= 6 \int \left\{ (t^2 - t + 1) - \frac{1}{1+t} \right\} dt \\ &= 6 \left[\left(\frac{t^3}{3} \right) - \left(\frac{t^2}{2} \right) + t - \log|1+t| \right] \\ &= 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6 \log \left(1 + x^{\frac{1}{6}} \right) + C \\ &= 2\sqrt{x} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6 \log \left(1 + x^{\frac{1}{6}} \right) + C \end{aligned}$$

Question 6:

$$\frac{5x}{(x+1)(x^2+9)}$$

Answer

$$\text{Let } \frac{5x}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9} \quad \dots(1)$$

$$\Rightarrow 5x = A(x^2+9) + (Bx+C)(x+1)$$

$$\Rightarrow 5x = Ax^2 + 9A + Bx^2 + Bx + Cx + C$$

Equating the coefficients of x^2 , x , and constant term, we obtain

$$A + B = 0$$

$$B + C = 5$$

$$9A + C = 0$$

On solving these equations, we obtain

$$A = -\frac{1}{2}, B = \frac{1}{2}, \text{ and } C = \frac{9}{2}$$

From equation (1), we obtain

$$\frac{5x}{(x+1)(x^2+9)} = \frac{-1}{2(x+1)} + \frac{\frac{x}{2} + \frac{9}{2}}{x^2+9}$$

$$\begin{aligned} \int \frac{5x}{(x+1)(x^2+9)} dx &= \int \left[\frac{-1}{2(x+1)} + \frac{(x+9)}{2(x^2+9)} \right] dx \\ &= -\frac{1}{2} \log|x+1| + \frac{1}{2} \int \frac{x}{x^2+9} dx + \frac{9}{2} \int \frac{1}{x^2+9} dx \\ &= -\frac{1}{2} \log|x+1| + \frac{1}{4} \int \frac{2x}{x^2+9} dx + \frac{9}{2} \int \frac{1}{x^2+9} dx \\ &= -\frac{1}{2} \log|x+1| + \frac{1}{4} \log|x^2+9| + \frac{9}{2} \cdot \frac{1}{3} \tan^{-1} \frac{x}{3} \\ &= -\frac{1}{2} \log|x+1| + \frac{1}{4} \log(x^2+9) + \frac{3}{2} \tan^{-1} \frac{x}{3} + C \end{aligned}$$

Question 7:

$$\frac{\sin x}{\sin(x-a)}$$

Answer

$$\frac{\sin x}{\sin(x-a)}$$

Let $x - a = t \Rightarrow dx = dt$

$$\begin{aligned} \int \frac{\sin x}{\sin(x-a)} dx &= \int \frac{\sin(t+a)}{\sin t} dt \\ &= \int \frac{\sin t \cos a + \cos t \sin a}{\sin t} dt \\ &= \int (\cos a + \cot t \sin a) dt \\ &= t \cos a + \sin a \log |\sin t| + C_1 \\ &= (x-a) \cos a + \sin a \log |\sin(x-a)| + C_1 \\ &= x \cos a + \sin a \log |\sin(x-a)| - a \cos a + C_1 \\ &= \sin a \log |\sin(x-a)| + x \cos a + C \end{aligned}$$

Question 8:

$$\frac{e^{5 \log x} - e^{4 \log x}}{e^{3 \log x} - e^{2 \log x}}$$

Answer

$$\begin{aligned} \frac{e^{5 \log x} - e^{4 \log x}}{e^{3 \log x} - e^{2 \log x}} &= \frac{e^{4 \log x} (e^{\log x} - 1)}{e^{2 \log x} (e^{\log x} - 1)} \\ &= e^{2 \log x} \\ &= e^{\log x^2} \\ &= x^2 \\ \therefore \int \frac{e^{5 \log x} - e^{4 \log x}}{e^{3 \log x} - e^{2 \log x}} dx &= \int x^2 dx = \frac{x^3}{3} + C \end{aligned}$$

Question 9:

$$\frac{\cos x}{\sqrt{4 - \sin^2 x}}$$

Answer

$$\frac{\cos x}{\sqrt{4 - \sin^2 x}}$$

Let $\sin x = t \Rightarrow \cos x \, dx = dt$

$$\begin{aligned} \Rightarrow \int \frac{\cos x}{\sqrt{4 - \sin^2 x}} dx &= \int \frac{dt}{\sqrt{(2)^2 - (t)^2}} \\ &= \sin^{-1} \left(\frac{t}{2} \right) + C \\ &= \sin^{-1} \left(\frac{\sin x}{2} \right) + C \end{aligned}$$

Question 10:

$$\frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x}$$

Answer

$$\begin{aligned} \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} &= \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{\sin^2 x + \cos^2 x - \sin^2 x \cos^2 x - \sin^2 x \cos^2 x} \\ &= \frac{(\sin^4 x + \cos^4 x)(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)}{(\sin^2 x - \sin^2 x \cos^2 x) + (\cos^2 x - \sin^2 x \cos^2 x)} \\ &= \frac{(\sin^4 x + \cos^4 x)(\sin^2 x - \cos^2 x)}{\sin^2 x(1 - \cos^2 x) + \cos^2 x(1 - \sin^2 x)} \\ &= \frac{-(\sin^4 x + \cos^4 x)(\cos^2 x - \sin^2 x)}{(\sin^4 x + \cos^4 x)} \\ &= -\cos 2x \end{aligned}$$

$$\therefore \int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx = \int -\cos 2x dx = -\frac{\sin 2x}{2} + C$$

Question 11:

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Answer

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Multiplying and dividing by $\sin(a-b)$, we obtain

$$\begin{aligned} & \frac{1}{\sin(a-b)} \left[\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)} \right] \\ &= \frac{1}{\sin(a-b)} \left[\frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)} \right] \\ &= \frac{1}{\sin(a-b)} \left[\frac{\sin(x+a) \cdot \cos(x+b) - \cos(x+a) \sin(x+b)}{\cos(x+a)\cos(x+b)} \right] \\ &= \frac{1}{\sin(a-b)} \left[\frac{\sin(x+a)}{\cos(x+a)} - \frac{\sin(x+b)}{\cos(x+b)} \right] \\ &= \frac{1}{\sin(a-b)} [\tan(x+a) - \tan(x+b)] \end{aligned}$$

$$\begin{aligned} \int \frac{1}{\cos(x+a)\cos(x+b)} dx &= \frac{1}{\sin(a-b)} \int [\tan(x+a) - \tan(x+b)] dx \\ &= \frac{1}{\sin(a-b)} [-\log|\cos(x+a)| + \log|\cos(x+b)|] + C \\ &= \frac{1}{\sin(a-b)} \log \left| \frac{\cos(x+b)}{\cos(x+a)} \right| + C \end{aligned}$$

Question 12:

$$\frac{x^3}{\sqrt{1-x^8}}$$

Answer

$$\frac{x^3}{\sqrt{1-x^8}}$$

Let $x^4 = t \Rightarrow 4x^3 dx = dt$

$$\begin{aligned}\Rightarrow \int \frac{x^3}{\sqrt{1-x^8}} dx &= \frac{1}{4} \int \frac{dt}{\sqrt{1-t^2}} \\ &= \frac{1}{4} \sin^{-1} t + C \\ &= \frac{1}{4} \sin^{-1}(x^4) + C\end{aligned}$$

Question 13:

$$\frac{e^x}{(1+e^x)(2+e^x)}$$

Answer

$$\frac{e^x}{(1+e^x)(2+e^x)}$$

Let $e^x = t \Rightarrow e^x dx = dt$

$$\begin{aligned}\Rightarrow \int \frac{e^x}{(1+e^x)(2+e^x)} dx &= \int \frac{dt}{(t+1)(t+2)} \\ &= \int \left[\frac{1}{(t+1)} - \frac{1}{(t+2)} \right] dt \\ &= \log|t+1| - \log|t+2| + C \\ &= \log \left| \frac{t+1}{t+2} \right| + C \\ &= \log \left| \frac{1+e^x}{2+e^x} \right| + C\end{aligned}$$

Question 14:

$$\frac{1}{(x^2+1)(x^2+4)}$$

Answer

$$\begin{aligned} \therefore \frac{1}{(x^2+1)(x^2+4)} &= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} \\ \Rightarrow 1 &= (Ax+B)(x^2+4) + (Cx+D)(x^2+1) \\ \Rightarrow 1 &= Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D \end{aligned}$$

Equating the coefficients of x^3 , x^2 , x , and constant term, we obtain

$$A + C = 0$$

$$B + D = 0$$

$$4A + C = 0$$

$$4B + D = 1$$

On solving these equations, we obtain

$$A = 0, B = \frac{1}{3}, C = 0, \text{ and } D = -\frac{1}{3}$$

From equation (1), we obtain

$$\begin{aligned} \frac{1}{(x^2+1)(x^2+4)} &= \frac{1}{3(x^2+1)} - \frac{1}{3(x^2+4)} \\ \int \frac{1}{(x^2+1)(x^2+4)} dx &= \frac{1}{3} \int \frac{1}{x^2+1} dx - \frac{1}{3} \int \frac{1}{x^2+4} dx \\ &= \frac{1}{3} \tan^{-1} x - \frac{1}{3} \cdot \frac{1}{2} \tan^{-1} \frac{x}{2} + C \\ &= \frac{1}{3} \tan^{-1} x - \frac{1}{6} \tan^{-1} \frac{x}{2} + C \end{aligned}$$

Question 15:

$$\cos^3 x e^{\log \sin x}$$

Answer

$$\cos^3 x e^{\log \sin x} = \cos^3 x \times \sin x$$

$$\text{Let } \cos x = t \Rightarrow -\sin x dx = dt$$

$$\begin{aligned}\Rightarrow \int \cos^3 x e^{\log \sin x} dx &= \int \cos^3 x \sin x dx \\ &= -\int t \cdot dt \\ &= -\frac{t^4}{4} + C \\ &= -\frac{\cos^4 x}{4} + C\end{aligned}$$

Question 16:

$$e^{3 \log x} (x^4 + 1)^{-1}$$

Answer

$$e^{3 \log x} (x^4 + 1)^{-1} = e^{\log x^3} (x^4 + 1)^{-1} = \frac{x^3}{(x^4 + 1)}$$

$$\text{Let } x^4 + 1 = t \Rightarrow 4x^3 dx = dt$$

$$\begin{aligned}\Rightarrow \int e^{3 \log x} (x^4 + 1)^{-1} dx &= \int \frac{x^3}{(x^4 + 1)} dx \\ &= \frac{1}{4} \int \frac{dt}{t} \\ &= \frac{1}{4} \log |t| + C \\ &= \frac{1}{4} \log |x^4 + 1| + C \\ &= \frac{1}{4} \log (x^4 + 1) + C\end{aligned}$$

Question 17:

$$f'(ax + b) [f(ax + b)]^n$$

Answer

$$f'(ax+b)[f(ax+b)]^n$$

$$\text{Let } f(ax+b) = t \Rightarrow af'(ax+b)dx = dt$$

$$\begin{aligned}\Rightarrow \int f'(ax+b)[f(ax+b)]^n dx &= \frac{1}{a} \int t^n dt \\ &= \frac{1}{a} \left[\frac{t^{n+1}}{n+1} \right] \\ &= \frac{1}{a(n+1)} (f(ax+b))^{n+1} + C\end{aligned}$$

Question 18:

$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$$

Answer

$$\begin{aligned} \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} &= \frac{1}{\sqrt{\sin^3 x (\sin x \cos \alpha + \cos x \sin \alpha)}} \\ &= \frac{1}{\sqrt{\sin^4 x \cos \alpha + \sin^3 x \cos x \sin \alpha}} \\ &= \frac{1}{\sin^2 x \sqrt{\cos \alpha + \cot x \sin \alpha}} \\ &= \frac{\operatorname{cosec}^2 x}{\sqrt{\cos \alpha + \cot x \sin \alpha}} \end{aligned}$$

Let $\cos \alpha + \cot x \sin \alpha = t \Rightarrow -\operatorname{cosec}^2 x \sin \alpha \, dx = dt$

$$\begin{aligned} \therefore \int \frac{1}{\sin^3 x \sin(x+\alpha)} dx &= \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos \alpha + \cot x \sin \alpha}} dx \\ &= \frac{-1}{\sin \alpha} \int \frac{dt}{\sqrt{t}} \\ &= \frac{-1}{\sin \alpha} [2\sqrt{t}] + C \\ &= \frac{-1}{\sin \alpha} [2\sqrt{\cos \alpha + \cot x \sin \alpha}] + C \\ &= \frac{-2}{\sin \alpha} \sqrt{\cos \alpha + \frac{\cos x \sin \alpha}{\sin x}} + C \\ &= \frac{-2}{\sin \alpha} \sqrt{\frac{\sin x \cos \alpha + \cos x \sin \alpha}{\sin x}} + C \\ &= -\frac{2}{\sin \alpha} \sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C \end{aligned}$$

Question 19:

$$\frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}}, x \in [0, 1]$$

Answer

$$\text{Let } I = \int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx$$

It is known that, $\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x} = \frac{\pi}{2}$

$$\begin{aligned} \Rightarrow I &= \int \frac{\left(\frac{\pi}{2} - \cos^{-1} \sqrt{x}\right) - \cos^{-1} \sqrt{x}}{\frac{\pi}{2}} dx \\ &= \frac{2\pi}{\pi} \int \left(\frac{1}{2} - 2 \cos^{-1} \sqrt{x}\right) dx \\ &= \frac{2\pi}{\pi} \cdot \frac{1}{2} \int 1 \cdot dx - \frac{4}{\pi} \int \cos^{-1} \sqrt{x} dx \\ &= x - \frac{4}{\pi} \int \cos^{-1} \sqrt{x} dx \quad \dots(1) \end{aligned}$$

$$\text{Let } I_1 = \int \cos^{-1} \sqrt{x} dx$$

Also, let $\sqrt{x} = t \Rightarrow dx = 2t dt$

$$\begin{aligned} \Rightarrow I_1 &= 2 \int \cos^{-1} t \cdot t dt \\ &= 2 \left[\cos^{-1} t \cdot \frac{t^2}{2} - \int \frac{-1}{\sqrt{1-t^2}} \cdot \frac{t^2}{2} dt \right] \\ &= t^2 \cos^{-1} t + \int \frac{t^2}{\sqrt{1-t^2}} dt \\ &= t^2 \cos^{-1} t - \int \frac{1-t^2-1}{\sqrt{1-t^2}} dt \\ &= t^2 \cos^{-1} t - \int \sqrt{1-t^2} dt + \int \frac{1}{\sqrt{1-t^2}} dt \\ &= t^2 \cos^{-1} t - \frac{t}{2} \sqrt{1-t^2} - \frac{1}{2} \sin^{-1} t + \sin^{-1} t \\ &= t^2 \cos^{-1} t - \frac{t}{2} \sqrt{1-t^2} + \frac{1}{2} \sin^{-1} t \end{aligned}$$

From equation (1), we obtain

$$\begin{aligned}
 I &= x - \frac{4}{\pi} \left[t^2 \cos t - \frac{t}{2} \sqrt{1-t^2} + \frac{1}{2} \sin^{-1} t \right] \\
 &= x - \frac{4}{\pi} \left[x \cos^{-1} \sqrt{x} - \frac{\sqrt{x}}{2} \sqrt{1-x} + \frac{1}{2} \sin^{-1} \sqrt{x} \right] \\
 &= x - \frac{4\pi}{\pi} \left[x \left(\frac{\pi}{2} - \sin^{-1} \sqrt{x} \right) - \frac{\sqrt{x-x^2}}{2} + \frac{1}{2} \sin^{-1} \sqrt{x} \right] \\
 &= x - 2x + \frac{4x}{\pi} \sin^{-1} \sqrt{x} + \frac{2}{\pi} \sqrt{x-x^2} - \frac{2}{\pi} \sin^{-1} \sqrt{x} \\
 &= -x + \frac{2}{\pi} \left[(2x-1) \sin^{-1} \sqrt{x} \right] + \frac{2}{\pi} \sqrt{x-x^2} + C \\
 &= \frac{2(2x-1)}{\pi} \sin^{-1} \sqrt{x} + \frac{2}{\pi} \sqrt{x-x^2} - x + C
 \end{aligned}$$

Question 20:

$$\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

Answer

$$I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} dx$$

$$\text{Let } x = \cos^2 \theta \Rightarrow dx = -2 \sin \theta \cos \theta d\theta$$

$$I = \int \sqrt{\frac{1-\cos \theta}{1+\cos \theta}} (-2 \sin \theta \cos \theta) d\theta$$

$$= - \int \sqrt{\frac{2 \sin^2 \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}}} \sin 2\theta d\theta$$

$$= - \int \tan \frac{\theta}{2} \cdot 2 \sin \theta \cos \theta d\theta$$

$$= -2 \int \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \left(2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right) \cos \theta d\theta$$

$$\begin{aligned}
 &= -4 \int \sin^2 \frac{\theta}{2} \cos \theta d\theta \\
 &= -4 \int \sin^2 \frac{\theta}{2} \cdot \left(2 \cos^2 \frac{\theta}{2} - 1 \right) d\theta \\
 &= -4 \int \left(2 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \right) d\theta \\
 &= -8 \int \sin^2 \frac{\theta}{2} \cdot \cos^2 \frac{\theta}{2} d\theta + 4 \int \sin^2 \frac{\theta}{2} d\theta \\
 &= -2 \int \sin^2 \theta d\theta + 4 \int \sin^2 \frac{\theta}{2} d\theta \\
 &= -2 \int \left(\frac{1 - \cos 2\theta}{2} \right) d\theta + 4 \int \frac{1 - \cos \theta}{2} d\theta \\
 &= -2 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right] + 4 \left[\frac{\theta}{2} - \frac{\sin \theta}{2} \right] + C \\
 &= -\theta + \frac{\sin 2\theta}{2} + 2\theta - 2 \sin \theta + C \\
 &= \theta + \frac{\sin 2\theta}{2} - 2 \sin \theta + C \\
 &= \theta + \frac{2 \sin \theta \cos \theta}{2} - 2 \sin \theta + C \\
 &= \theta + \sqrt{1 - \cos^2 \theta} \cdot \cos \theta - 2\sqrt{1 - \cos^2 \theta} + C \\
 &= \cos^{-1} \sqrt{x} + \sqrt{1-x} \cdot \sqrt{x} - 2\sqrt{1-x} + C \\
 &= -2\sqrt{1-x} + \cos^{-1} \sqrt{x} + \sqrt{x(1-x)} + C \\
 &= -2\sqrt{1-x} + \cos^{-1} \sqrt{x} + \sqrt{x-x^2} + C
 \end{aligned}$$

Question 21:

$$\frac{2 + \sin 2x}{1 + \cos 2x} e^x$$

Answer

$$\begin{aligned}
 I &= \int \left(\frac{2 + \sin 2x}{1 + \cos 2x} \right) e^x \\
 &= \int \left(\frac{2 + 2 \sin x \cos x}{2 \cos^2 x} \right) e^x \\
 &= \int \left(\frac{1 + \sin x \cos x}{\cos^2 x} \right) e^x \\
 &= \int (\sec^2 x + \tan x) e^x
 \end{aligned}$$

Let $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$

$$\begin{aligned}
 \therefore I &= \int [f(x) + f'(x)] e^x dx \\
 &= e^x f(x) + C \\
 &= e^x \tan x + C
 \end{aligned}$$

Question 22:

$$\frac{x^2 + x + 1}{(x+1)^2(x+2)}$$

Answer

$$\text{Let } \frac{x^2 + x + 1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2} \quad \dots(1)$$

$$\Rightarrow x^2 + x + 1 = A(x+1)(x+2) + B(x+2) + C(x^2 + 2x + 1)$$

$$\Rightarrow x^2 + x + 1 = A(x^2 + 3x + 2) + B(x+2) + C(x^2 + 2x + 1)$$

$$\Rightarrow x^2 + x + 1 = (A+C)x^2 + (3A+B+2C)x + (2A+2B+C)$$

Equating the coefficients of x^2 , x , and constant term, we obtain

$$A + C = 1$$

$$3A + B + 2C = 1$$

$$2A + 2B + C = 1$$

On solving these equations, we obtain

$$A = -2, B = 1, \text{ and } C = 3$$

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{-2}{(x+1)} + \frac{3}{(x+2)} + \frac{1}{(x+1)^2}$$

$$\int \frac{x^2+x+1}{(x+1)^2(x+2)} dx = -2 \int \frac{1}{x+1} dx + 3 \int \frac{1}{(x+2)} dx + \int \frac{1}{(x+1)^2} dx$$

$$= -2 \log|x+1| + 3 \log|x+2| - \frac{1}{(x+1)} + C$$

From equation (1), we obtain

Question 23:

$$\tan^{-1} \sqrt{\frac{1-x}{1+x}}$$

Answer

$$I = \int \tan^{-1} \sqrt{\frac{1-x}{1+x}} dx$$

$$\text{Let } x = \cos \theta \Rightarrow dx = -\sin \theta d\theta$$

$$I = \int \tan^{-1} \sqrt{\frac{1-\cos \theta}{1+\cos \theta}} (-\sin \theta d\theta)$$

$$= -\int \tan^{-1} \sqrt{\frac{2 \sin^2 \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}}} \sin \theta d\theta$$

$$= -\int \tan^{-1} \tan \frac{\theta}{2} \cdot \sin \theta d\theta$$

$$= -\frac{1}{2} \int \theta \cdot \sin \theta d\theta$$

$$= -\frac{1}{2} \left[\theta \cdot (-\cos \theta) - \int 1 \cdot (-\cos \theta) d\theta \right]$$

$$= -\frac{1}{2} \left[-\theta \cos \theta + \sin \theta \right]$$

$$= +\frac{1}{2} \theta \cos \theta - \frac{1}{2} \sin \theta$$

$$= \frac{1}{2} \cos^{-1} x \cdot x - \frac{1}{2} \sqrt{1-x^2} + C$$

$$= \frac{x}{2} \cos^{-1} x - \frac{1}{2} \sqrt{1-x^2} + C$$

$$= \frac{1}{2} \left(x \cos^{-1} x - \sqrt{1-x^2} \right) + C$$

Question 24:

$$\frac{\sqrt{x^2+1} [\log(x^2+1) - 2 \log x]}{x^4}$$

Answer

$$\begin{aligned} \frac{\sqrt{x^2+1} [\log(x^2+1) - 2 \log x]}{x^4} &= \frac{\sqrt{x^2+1}}{x^4} [\log(x^2+1) - \log x^2] \\ &= \frac{\sqrt{x^2+1}}{x^4} \left[\log \left(\frac{x^2+1}{x^2} \right) \right] \\ &= \frac{\sqrt{x^2+1}}{x^4} \log \left(1 + \frac{1}{x^2} \right) \\ &= \frac{1}{x^3} \sqrt{\frac{x^2+1}{x^2}} \log \left(1 + \frac{1}{x^2} \right) \\ &= \frac{1}{x^3} \sqrt{1 + \frac{1}{x^2}} \log \left(1 + \frac{1}{x^2} \right) \end{aligned}$$

$$\text{Let } 1 + \frac{1}{x^2} = t \Rightarrow \frac{-2}{x^3} dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{1}{x^3} \sqrt{1 + \frac{1}{x^2}} \log \left(1 + \frac{1}{x^2} \right) dx \\ &= -\frac{1}{2} \int \sqrt{t} \log t \, dt \\ &= -\frac{1}{2} \int t^{\frac{1}{2}} \cdot \log t \, dt \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
 I &= -\frac{1}{2} \left[\log t \cdot \int t^{\frac{1}{2}} dt - \left\{ \left(\frac{d}{dt} \log t \right) \int t^{\frac{1}{2}} dt \right\} dt \right] \\
 &= -\frac{1}{2} \left[\log t \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \int \frac{1}{t} \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} dt \right] \\
 &= -\frac{1}{2} \left[\frac{2}{3} t^{\frac{3}{2}} \log t - \frac{2}{3} \int t^{\frac{1}{2}} dt \right] \\
 &= -\frac{1}{2} \left[\frac{2}{3} t^{\frac{3}{2}} \log t - \frac{4}{9} t^{\frac{3}{2}} \right] \\
 &= -\frac{1}{3} t^{\frac{3}{2}} \log t + \frac{2}{9} t^{\frac{3}{2}} \\
 &= -\frac{1}{3} t^{\frac{3}{2}} \left[\log t - \frac{2}{3} \right] \\
 &= -\frac{1}{3} \left(1 + \frac{1}{x^2} \right)^{\frac{3}{2}} \left[\log \left(1 + \frac{1}{x^2} \right) - \frac{2}{3} \right] + C
 \end{aligned}$$

Question 25:

$$\int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx$$

Answer

$$\begin{aligned}
 I &= \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx \\
 &= \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1 - 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \sin^2 \frac{x}{2}} \right) dx \\
 &= \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{\operatorname{cosec}^2 \frac{x}{2}}{2} - \cot \frac{x}{2} \right) dx
 \end{aligned}$$

Let $f(x) = -\cot \frac{x}{2}$

$$\Rightarrow f'(x) = -\left(-\frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} \right) = \frac{1}{2} \operatorname{cosec}^2 \frac{x}{2}$$

$$\therefore I = \int_{\frac{\pi}{2}}^{\pi} e^x (f(x) + f'(x)) dx$$

$$= [e^x \cdot f(x) dx]_{\frac{\pi}{2}}^{\pi}$$

$$= - \left[e^x \cdot \cot \frac{x}{2} \right]_{\frac{\pi}{2}}^{\pi}$$

$$= - \left[e^{\pi} \times \cot \frac{\pi}{2} - e^{\frac{\pi}{2}} \times \cot \frac{\pi}{4} \right]$$

$$= - \left[e^{\pi} \times 0 - e^{\frac{\pi}{2}} \times 1 \right]$$

$$= e^{\frac{\pi}{2}}$$

Question 26:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\frac{(\sin x \cos x)}{\cos^4 x}}{(\cos^4 x + \sin^4 x)} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx$$

$$\text{Let } \tan^2 x = t \Rightarrow 2 \tan x \sec^2 x dx = dt$$

$$\text{when } x = \frac{\pi}{4}, t = 1$$

When $x = 0$, $t = 0$ and

$$\therefore I = \frac{1}{2} \int_0^1 \frac{dt}{1+t^2}$$

$$= \frac{1}{2} [\tan^{-1} t]_0^1$$

$$= \frac{1}{2} [\tan^{-1} 1 - \tan^{-1} 0]$$

$$= \frac{1}{2} \left[\frac{\pi}{4} \right]$$

$$= \frac{\pi}{8}$$

Question 27:

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x dx}{\cos^2 x + 4 \sin^2 x}$$

Answer

$$\begin{aligned}
 \text{Let } I &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 \sin^2 x} dx \\
 \Rightarrow I &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4(1 - \cos^2 x)} dx \\
 \Rightarrow I &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 - 4 \cos^2 x} dx \\
 \Rightarrow I &= \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 3 \cos^2 x - 4}{4 - 3 \cos^2 x} dx \\
 \Rightarrow I &= \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 3 \cos^2 x}{4 - 3 \cos^2 x} dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4}{4 - 3 \cos^2 x} dx \\
 \Rightarrow I &= \frac{-1}{3} \int_0^{\frac{\pi}{2}} 1 dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4 \sec^2 x}{4 \sec^2 x - 3} dx \\
 \Rightarrow I &= \frac{-1}{3} [x]_0^{\frac{\pi}{2}} + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4 \sec^2 x}{4(1 + \tan^2 x) - 3} dx \\
 \Rightarrow I &= -\frac{\pi}{6} + \frac{2}{3} \int_0^{\frac{\pi}{2}} \frac{2 \sec^2 x}{1 + 4 \tan^2 x} dx \quad \dots(1)
 \end{aligned}$$

Consider, $\int_0^{\frac{\pi}{2}} \frac{2 \sec^2 x}{1 + 4 \tan^2 x} dx$

Let $2 \tan x = t \Rightarrow 2 \sec^2 x dx = dt$

When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = \infty$

$$\begin{aligned}
 \Rightarrow \int_0^{\frac{\pi}{2}} \frac{2 \sec^2 x}{1 + 4 \tan^2 x} dx &= \int_0^{\infty} \frac{dt}{1 + t^2} \\
 &= [\tan^{-1} t]_0^{\infty} \\
 &= [\tan^{-1}(\infty) - \tan^{-1}(0)] \\
 &= \frac{\pi}{2}
 \end{aligned}$$

Therefore, from (1), we obtain

$$I = -\frac{\pi}{6} + \frac{2}{3} \left[\frac{\pi}{2} \right] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

Question 28:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

Answer

$$\text{Let } I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{-(-\sin 2x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-1+1-2 \sin x \cos x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{1-(\sin^2 x + \cos^2 x - 2 \sin x \cos x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x) dx}{\sqrt{1-(\sin x - \cos x)^2}}$$

$$\text{Let } (\sin x - \cos x) = t \Rightarrow (\sin x + \cos x) dx = dt$$

$$x = \frac{\pi}{6}, t = \left(\frac{1-\sqrt{3}}{2}\right) \quad x = \frac{\pi}{3}, t = \left(\frac{\sqrt{3}-1}{2}\right)$$

When $x = \frac{\pi}{6}, t = \left(\frac{1-\sqrt{3}}{2}\right)$ and when $x = \frac{\pi}{3}, t = \left(\frac{\sqrt{3}-1}{2}\right)$

$$I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$\Rightarrow I = \int_{\left(\frac{1-\sqrt{3}}{2}\right)}^{\left(\frac{\sqrt{3}-1}{2}\right)} \frac{dt}{\sqrt{1-t^2}}$$

As $\frac{1}{\sqrt{1-(-t)^2}} = \frac{1}{\sqrt{1-t^2}}$, therefore, $\frac{1}{\sqrt{1-t^2}}$ is an even function.

It is known that if $f(x)$ is an even function, then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

$$\begin{aligned} \Rightarrow I &= 2 \int_0^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}} \\ &= \left[2 \sin^{-1} t \right]_0^{\frac{\sqrt{3}-1}{2}} \\ &= 2 \sin^{-1} \left(\frac{\sqrt{3}-1}{2} \right) \end{aligned}$$

Question 29:

$$\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$$

Answer

$$\begin{aligned} \text{Let } I &= \int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}} \\ I &= \int_0^1 \frac{1}{(\sqrt{1+x}-\sqrt{x})} \times \frac{(\sqrt{1+x}+\sqrt{x})}{(\sqrt{1+x}+\sqrt{x})} dx \\ &= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x} dx \\ &= \int_0^1 \sqrt{1+x} dx + \int_0^1 \sqrt{x} dx \\ &= \left[\frac{2}{3}(1+x)^{\frac{3}{2}} \right]_0^1 + \left[\frac{2}{3}(x)^{\frac{3}{2}} \right]_0^1 \\ &= \frac{2}{3} \left[(2)^{\frac{3}{2}} - 1 \right] + \frac{2}{3} [1] \\ &= \frac{2}{3} (2)^{\frac{3}{2}} \\ &= \frac{2 \cdot 2\sqrt{2}}{3} \\ &= \frac{4\sqrt{2}}{3} \end{aligned}$$

Question 30:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$$

$$\text{Also, let } \sin x - \cos x = t \Rightarrow (\cos x + \sin x) dx = dt$$

$$\text{When } x = 0, t = -1 \text{ and when } x = \frac{\pi}{4}, t = 0$$

$$\Rightarrow (\sin x - \cos x)^2 = t^2$$

$$\Rightarrow \sin^2 x + \cos^2 x - 2 \sin x \cos x = t^2$$

$$\Rightarrow 1 - \sin 2x = t^2$$

$$\Rightarrow \sin 2x = 1 - t^2$$

$$\begin{aligned} \therefore I &= \int_{-1}^0 \frac{dt}{9 + 16(1 - t^2)} \\ &= \int_{-1}^0 \frac{dt}{9 + 16 - 16t^2} \\ &= \int_{-1}^0 \frac{dt}{25 - 16t^2} = \int_{-1}^0 \frac{dt}{(5)^2 - (4t)^2} \\ &= \frac{1}{4} \left[\frac{1}{2(5)} \log \left| \frac{5+4t}{5-4t} \right| \right]_{-1}^0 \\ &= \frac{1}{40} \left[\log(1) - \log \left| \frac{1}{9} \right| \right] \\ &= \frac{1}{40} \log 9 \end{aligned}$$

Question 31:

$$\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx = \int_0^{\frac{\pi}{2}} 2 \sin x \cos x \tan^{-1}(\sin x) dx$$

$$\text{Also, let } \sin x = t \Rightarrow \cos x dx = dt$$

$$\text{When } x = 0, t = 0 \text{ and when } x = \frac{\pi}{2}, t = 1$$

$$\Rightarrow I = 2 \int_0^1 t \tan^{-1}(t) dt \quad \dots(1)$$

$$\text{Consider } \int t \cdot \tan^{-1} t dt = \tan^{-1} t \cdot \int t dt - \int \left\{ \frac{d}{dt}(\tan^{-1} t) \int t dt \right\} dt$$

$$= \tan^{-1} t \cdot \frac{t^2}{2} - \int \frac{1}{1+t^2} \cdot \frac{t^2}{2} dt$$

$$= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2} \int \frac{t^2 + 1 - 1}{1+t^2} dt$$

$$= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2} \int 1 dt + \frac{1}{2} \int \frac{1}{1+t^2} dt$$

$$= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2} \cdot t + \frac{1}{2} \tan^{-1} t$$

$$\Rightarrow \int_0^1 t \cdot \tan^{-1} t dt = \left[\frac{t^2 \cdot \tan^{-1} t}{2} - \frac{t}{2} + \frac{1}{2} \tan^{-1} t \right]_0^1$$

$$= \frac{1}{2} \left[\frac{\pi}{4} - 1 + \frac{\pi}{4} \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{2} - 1 \right] = \frac{\pi}{4} - \frac{1}{2}$$

From equation (1), we obtain

$$I = 2 \left[\frac{\pi}{4} - \frac{1}{2} \right] = \frac{\pi}{2} - 1$$

Question 32:

$$\int_0^{\frac{\pi}{2}} \frac{x \tan x}{\sec x + \tan x} dx$$

Answer

$$\text{Let } I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx \quad \dots(1)$$

$$I = \int_0^{\pi} \left\{ \frac{(\pi - x) \tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} \right\} dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a - x) dx \right)$$

$$\Rightarrow I = \int_0^{\pi} \left\{ \frac{-(\pi - x) \tan x}{-(\sec x + \tan x)} \right\} dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x) \tan x}{\sec x + \tan x} dx \quad \dots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x + \tan x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x + 1 - 1}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} 1 \cdot dx - \pi \int_0^{\pi} \frac{1}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi [x]_0^{\pi} - \pi \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi^2 - \pi \int_0^{\pi} (\sec^2 x - \tan x \sec x) dx$$

$$\Rightarrow 2I = \pi^2 - \pi [\tan x - \sec x]_0^{\pi}$$

$$\Rightarrow 2I = \pi^2 - \pi [\tan \pi - \sec \pi - \tan 0 + \sec 0]$$

$$\Rightarrow 2I = \pi^2 - \pi [0 - (-1) - 0 + 1]$$

$$\Rightarrow 2I = \pi^2 - 2\pi$$

$$\Rightarrow 2I = \pi(\pi - 2)$$

$$\Rightarrow I = \frac{\pi}{2}(\pi - 2)$$

$$\int_1^4 [|x-1| + |x-2| + |x-3|] dx$$

Answer

$$\text{Let } I = \int_1^4 [|x-1| + |x-2| + |x-3|] dx$$

$$\Rightarrow I = \int_1^4 |x-1| dx + \int_1^4 |x-2| dx + \int_1^4 |x-3| dx$$

$$I = I_1 + I_2 + I_3 \quad \dots(1)$$

$$\text{where, } I_1 = \int_1^4 |x-1| dx, I_2 = \int_1^4 |x-2| dx, \text{ and } I_3 = \int_1^4 |x-3| dx$$

$$I_1 = \int_1^4 |x-1| dx$$

$$(x-1) \geq 0 \text{ for } 1 \leq x \leq 4$$

$$\therefore I_1 = \int_1^4 (x-1) dx$$

$$\Rightarrow I_1 = \left[\frac{x^2}{2} - x \right]_1^4$$

$$\Rightarrow I_1 = \left[8 - 4 - \frac{1}{2} + 1 \right] = \frac{9}{2} \quad \dots(2)$$

$$I_2 = \int_1^4 |x-2| dx$$

$$x-2 \geq 0 \text{ for } 2 \leq x \leq 4 \text{ and } x-2 \leq 0 \text{ for } 1 \leq x \leq 2$$

$$\therefore I_2 = \int_1^2 (2-x) dx + \int_2^4 (x-2) dx$$

$$\Rightarrow I_2 = \left[2x - \frac{x^2}{2} \right]_1^2 + \left[\frac{x^2}{2} - 2x \right]_2^4$$

$$\Rightarrow I_2 = \left[4 - 2 - 2 + \frac{1}{2} \right] + [8 - 8 - 2 + 4]$$

$$\Rightarrow I_2 = \frac{1}{2} + 2 = \frac{5}{2} \quad \dots(3)$$

Question 33:

$$\begin{aligned}
 I_3 &= \int_1^4 |x-3| dx \\
 x-3 &\geq 0 \text{ for } 3 \leq x \leq 4 \text{ and } x-3 \leq 0 \text{ for } 1 \leq x \leq 3 \\
 \therefore I_3 &= \int_1^3 (3-x) dx + \int_3^4 (x-3) dx \\
 \Rightarrow I_3 &= \left[3x - \frac{x^2}{2} \right]_1^3 + \left[\frac{x^2}{2} - 3x \right]_3^4 \\
 \Rightarrow I_3 &= \left[9 - \frac{9}{2} - 3 + \frac{1}{2} \right] + \left[8 - 12 - \frac{9}{2} + 9 \right] \\
 \Rightarrow I_3 &= [6-4] + \left[\frac{1}{2} \right] = \frac{5}{2} \quad \dots(4)
 \end{aligned}$$

From equations (1), (2), (3), and (4), we obtain

$$I = \frac{9}{2} + \frac{5}{2} + \frac{5}{2} = \frac{19}{2}$$

Question 34:

$$\int \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log \frac{2}{3}$$

Answer

$$\text{Let } I = \int \frac{dx}{x^2(x+1)}$$

$$\text{Also, let } \frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\Rightarrow 1 = Ax(x+1) + B(x+1) + C(x^2)$$

$$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$$

Equating the coefficients of x^2 , x , and constant term, we obtain

$$A + C = 0$$

$$+ B = 0$$

$$B = 1$$

On solving these equations, we obtain

$$A = -1, C = 1, \text{ and } B = 1$$

$$\therefore \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)}$$

$$\Rightarrow I = \int_1^3 \left\{ -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)} \right\} dx$$

$$= \left[-\log x - \frac{1}{x} + \log(x+1) \right]_1^3$$

$$= \left[\log\left(\frac{x+1}{x}\right) - \frac{1}{x} \right]_1^3$$

$$= \log\left(\frac{4}{3}\right) - \frac{1}{3} - \log\left(\frac{2}{1}\right) + 1$$

$$= \log 4 - \log 3 - \log 2 + \frac{2}{3}$$

$$= \log 2 - \log 3 + \frac{2}{3}$$

$$= \log\left(\frac{2}{3}\right) + \frac{2}{3}$$

Hence, the given result is proved.

Question 35:

$$\int_0^1 xe^x dx = 1$$

Answer

$$\text{Let } I = \int_0^1 xe^x dx$$

Integrating by parts, we obtain

$$\begin{aligned} I &= x \int_0^1 e^x dx - \int_0^1 \left\{ \left(\frac{d}{dx}(x) \right) \int e^x dx \right\} dx \\ &= [xe^x]_0^1 - \int_0^1 e^x dx \\ &= [xe^x]_0^1 - [e^x]_0^1 \\ &= e - e + 1 \\ &= 1 \end{aligned}$$

Hence, the given result is proved.

Question 36:

$$\int_{-1}^1 x^{17} \cos^4 x dx = 0$$

Answer

$$\text{Let } I = \int_{-1}^1 x^{17} \cos^4 x dx$$

$$\text{Also, let } f(x) = x^{17} \cos^4 x$$

$$\Rightarrow f(-x) = (-x)^{17} \cos^4(-x) = -x^{17} \cos^4 x = -f(x)$$

Therefore, $f(x)$ is an odd function.

It is known that if $f(x)$ is an odd function, then $\int_{-a}^a f(x) dx = 0$

$$\therefore I = \int_{-1}^1 x^{17} \cos^4 x dx = 0$$

Hence, the given result is proved.

Question 37:

$$\int_0^{\frac{\pi}{2}} \sin^3 x \, dx = \frac{2}{3}$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \sin^3 x \, dx$$

$$I = \int_0^{\frac{\pi}{2}} \sin^2 x \cdot \sin x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \sin x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin x \, dx - \int_0^{\frac{\pi}{2}} \cos^2 x \cdot \sin x \, dx$$

$$= [-\cos x]_0^{\frac{\pi}{2}} + \left[\frac{\cos^3 x}{3} \right]_0^{\frac{\pi}{2}}$$

$$= 1 + \frac{1}{3}[-1] = 1 - \frac{1}{3} = \frac{2}{3}$$

Hence, the given result is proved.

Question 38:

$$\int_0^{\frac{\pi}{4}} 2 \tan^3 x dx = 1 - \log 2$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} 2 \tan^3 x dx$$

$$I = 2 \int_0^{\frac{\pi}{4}} \tan^2 x \tan x dx = 2 \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) \tan x dx$$

$$= 2 \int_0^{\frac{\pi}{4}} \sec^2 x \tan x dx - 2 \int_0^{\frac{\pi}{4}} \tan x dx$$

$$= 2 \left[\frac{\tan^2 x}{2} \right]_0^{\frac{\pi}{4}} + 2 \left[\log \cos x \right]_0^{\frac{\pi}{4}}$$

$$= 1 + 2 \left[\log \cos \frac{\pi}{4} - \log \cos 0 \right]$$

$$= 1 + 2 \left[\log \frac{1}{\sqrt{2}} - \log 1 \right]$$

$$= 1 - \log 2 - \log 1 = 1 - \log 2$$

Hence, the given result is proved.

Question 39:

$$\int_0^1 \sin^{-1} x dx = \frac{\pi}{2} - 1$$

Answer

$$\text{Let } I = \int_0^1 \sin^{-1} x dx$$

$$\Rightarrow I = \int_0^1 \sin^{-1} x \cdot 1 \cdot dx$$

Integrating by parts, we obtain

$$\begin{aligned} I &= \left[\sin^{-1} x \cdot x \right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} \cdot x \, dx \\ &= \left[x \sin^{-1} x \right]_0^1 + \frac{1}{2} \int_0^1 \frac{(-2x)}{\sqrt{1-x^2}} \, dx \end{aligned}$$

Let $1 - x^2 = t \Rightarrow -2x \, dx = dt$

When $x = 0$, $t = 1$ and when $x = 1$, $t = 0$

$$\begin{aligned} I &= \left[x \sin^{-1} x \right]_0^1 + \frac{1}{2} \int_1^0 \frac{dt}{\sqrt{t}} \\ &= \left[x \sin^{-1} x \right]_0^1 + \frac{1}{2} \left[2\sqrt{t} \right]_1^0 \\ &= \sin^{-1}(1) + \left[-\sqrt{1} \right] \\ &= \frac{\pi}{2} - 1 \end{aligned}$$

Hence, the given result is proved.

Question 40:

Evaluate $\int_0^1 e^{2-3x} dx$ as a limit of a sum.

Answer

$$\text{Let } I = \int_0^1 e^{2-3x} dx$$

It is known that,

$$\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + \dots + f(a+(n-1)h)]$$

$$\text{Where, } h = \frac{b-a}{n}$$

Here, $a = 0, b = 1$, and $f(x) = e^{2-3x}$

$$\Rightarrow h = \frac{1-0}{n} = \frac{1}{n}$$

$$\begin{aligned} \therefore \int_0^1 e^{2-3x} dx &= (1-0) \lim_{n \rightarrow \infty} \frac{1}{n} [f(0) + f(0+h) + \dots + f(0+(n-1)h)] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} [e^2 + e^{2-3h} + \dots + e^{2-3(n-1)h}] \end{aligned}$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[e^2 \left\{ 1 + e^{-3h} + e^{-6h} + e^{-9h} + \dots + e^{-3(n-1)h} \right\} \right] \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[e^2 \left\{ \frac{1 - (e^{-3h})^n}{1 - e^{-3h}} \right\} \right] \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[e^2 \left\{ \frac{1 - e^{-\frac{3}{n} \times n}}{1 - e^{-\frac{3}{n}}} \right\} \right] \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{e^2 (1 - e^{-3})}{1 - e^{-\frac{3}{n}}} \right] \\
 &= e^2 (e^{-3} - 1) \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{1}{e^{-\frac{3}{n}} - 1} \right] \\
 &= e^2 (e^{-3} - 1) \lim_{n \rightarrow \infty} \left(-\frac{1}{3} \right) \left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}} - 1} \right] \\
 &= \frac{-e^2 (e^{-3} - 1)}{3} \lim_{n \rightarrow \infty} \left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}} - 1} \right] \\
 &= \frac{-e^2 (e^{-3} - 1)}{3} (1) \quad \left[\lim_{n \rightarrow \infty} \frac{x}{e^x - 1} \right] \\
 &= \frac{-e^{-1} + e^2}{3} \\
 &= \frac{1}{3} \left(e^2 - \frac{1}{e} \right)
 \end{aligned}$$

Question 41:

$\int \frac{dx}{e^x + e^{-x}}$ is equal to

- A. $\tan^{-1}(e^x) + C$
- B. $\tan^{-1}(e^{-x}) + C$
- C. $\log(e^x - e^{-x}) + C$
- D. $\log(e^x + e^{-x}) + C$

Answer

$$\text{Let } I = \int \frac{dx}{e^x + e^{-x}} dx = \int \frac{e^x}{e^{2x} + 1} dx$$

Also, let $e^x = t \Rightarrow e^x dx = dt$

$$\begin{aligned} \therefore I &= \int \frac{dt}{1+t^2} \\ &= \tan^{-1} t + C \\ &= \tan^{-1}(e^x) + C \end{aligned}$$

Hence, the correct Answer is A.

Question 42:

$\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$ is equal to

- A. $\frac{-1}{\sin x + \cos x} + C$
B. $\log|\sin x + \cos x| + C$
C. $\log|\sin x - \cos x| + C$
D. $\frac{1}{(\sin x + \cos x)^2}$

Answer

$$\text{Let } I = \frac{\cos 2x}{(\cos x + \sin x)^2}$$

$$\begin{aligned} I &= \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx \\ &= \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\cos x + \sin x)^2} dx \\ &= \int \frac{\cos x - \sin x}{\cos x + \sin x} dx \end{aligned}$$

$$\text{Let } \cos x + \sin x = t \Rightarrow (\cos x - \sin x) dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{dt}{t} \\ &= \log|t| + C \\ &= \log|\cos x + \sin x| + C \end{aligned}$$

Hence, the correct Answer is B.

Question 43:

If $f(a+b-x) = f(x)$, then $\int_a^b x f(x) dx$ is equal to

A. $\frac{a+b}{2} \int_a^b f(b-x) dx$

B. $\frac{a+b}{2} \int_a^b f(b+x) dx$

C. $\frac{b-a}{2} \int_a^b f(x) dx$

D. $\frac{a+b}{2} \int_a^b f(x) dx$

Answer

Let $I = \int_a^b x f(x) dx \quad \dots(1)$

$I = \int_a^b (a+b-x) f(a+b-x) dx \quad \left(\int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right)$

$\Rightarrow I = \int_a^b (a+b-x) f(x) dx$

$\Rightarrow I = (a+b) \int_a^b f(x) dx - I \quad \text{[Using (1)]}$

$\Rightarrow I + I = (a+b) \int_a^b f(x) dx$

$\Rightarrow 2I = (a+b) \int_a^b f(x) dx$

$\Rightarrow I = \left(\frac{a+b}{2} \right) \int_a^b f(x) dx$

Hence, the correct Answer is D.

Question 44:

The value of $\int_b^1 \tan^{-1} \left(\frac{2x-1}{1+x-x^2} \right) dx$ is

A. 1

B. 0

C. - 1

D. $\frac{\pi}{4}$

Answer

$$\text{Let } I = \int_0^1 \tan^{-1} \left(\frac{2x-1}{1+x-x^2} \right) dx$$

$$\Rightarrow I = \int_0^1 \tan^{-1} \left(\frac{x-(1-x)}{1+x(1-x)} \right) dx$$

$$\Rightarrow I = \int_0^1 [\tan^{-1} x - \tan^{-1} (1-x)] dx \quad \dots(1)$$

$$\Rightarrow I = \int_0^1 [\tan^{-1} (1-x) - \tan^{-1} (1-1+x)] dx$$

$$\Rightarrow I = \int_0^1 [\tan^{-1} (1-x) - \tan^{-1} (x)] dx$$

$$\Rightarrow I = \int_0^1 [\tan^{-1} (1-x) - \tan^{-1} (x)] dx \quad \dots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^1 (\tan^{-1} x + \tan^{-1} (1-x) - \tan^{-1} (1-x) - \tan^{-1} x) dx$$

$$\Rightarrow 2I = 0$$

$$\Rightarrow I = 0$$

Hence, the correct Answer is B.