Exercise 5.1

Question 1:

Prove that the function $f(x) = 5x - 3$ is continuous at

Answer

The given function is $f(x) = 5x - 3$

At $x = 0$, $f(0) = 5 \times 0 - 3 = 3$

$\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} (5x - 3) = 5 \times 0 - 3 = -3$

$\therefore \lim\limits_{x \to 0} f(x) = f(0)$

Therefore, f is continuous at x = 0

At $x = -3$, $f(-3) = 5 \times (-3) - 3 = -18$

$\lim\limits_{x \to -3} f(x) = \lim\limits_{x \to -3} (5x - 3) = 5 \times (-3) - 3 = -18$

$\therefore \lim\limits_{x \to -3} f(x) = f(-3)$

Therefore, f is continuous at x = −3

At $x = 5$, $f(x) = f(5) = 5 \times 5 - 3 = 25 - 3 = 22$

$\lim\limits_{x \to 5} f(x) = \lim\limits_{x \to 5} (5x - 3) = 5 \times 5 - 3 = 22$

$\therefore \lim\limits_{x \to 5} f(x) = f(5)$

Therefore, f is continuous at x = 5

$x = 0$, at $x = -3$ and at $x = 5$.

Question 2:

Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.

Answer

The given function is $f(x) = 2x^2 - 1$

At $x = 3, f(x) = f(3) = 2 \times 3^2 - 1 = 17$

$\lim_{x \to 3} f(x) = \lim_{x \to 3}(2x^2 - 1) = 2 \times 3^2 - 1 = 17$

$\therefore \lim_{x \to 3} f(x) = f(3)$

Thus, f is continuous at x = 3

Question 3:

Examine the following functions for continuity.

(a) $f(x) = x - 5$ (b) $f(x) = \frac{1}{x-5}, x \neq 5$

(c) $f(x) = \frac{x^2 - 25}{x+5}, x \neq -5$ (d) $f(x) = |x - 5|$

Answer

(a) The given function is $f(x) = x - 5$

It is evident that f is defined at every real number k and its value at k is k − 5.

It is also observed that, $\lim_{x \to k} f(x) = \lim_{x \to k}(x - 5) = k - 5 = f(k)$

$\therefore \lim_{x \to k} f(x) = f(k)$

Hence, f is continuous at every real number and therefore, it is a continuous function.

(b) The given function is $f(x)=\frac{1}{x-5}, x \neq 5$

For any real number k ≠ 5, we obtain

$$\lim_{x\to k} f(x)=\lim_{x\to k}\frac{1}{x-5}=\frac{1}{k-5}$$

$$\text{Also, } f(k)=\frac{1}{k-5} \quad (\text{As } k\neq 5)$$

$$\therefore \lim_{x\to k} f(x)=f(k)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

(c) The given function is $f(x)=\frac{x^2-25}{x+5}, x\neq -5$

For any real number c ≠ −5, we obtain

$$\lim_{x\to c} f(x)=\lim_{x\to c}\frac{x^2-25}{x+5}=\lim_{x\to c}\frac{(x+5)(x-5)}{x+5}=\lim_{x\to c}(x-5)=(c-5)$$

$$\text{Also, } f(c)=\frac{(c+5)(c-5)}{c+5}=(c-5) \quad (\text{as } c\neq -5)$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

(d) The given function is $f(x)=|x-5|=\begin{cases}5-x, & \text{if } x<5\\ x-5, & \text{if } x\geq 5\end{cases}$

This function f is defined at all points of the real line.

Let c be a point on a real line. Then, c < 5 or c = 5 or c > 5

Case I: c < 5

Then, f (c) = 5 − c

$$\lim_{x\to c} f(x) = \lim_{x\to c}(5-x) = 5-c$$
$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all real numbers less than 5.

Case II : c = 5

Then, $f(c) = f(5) = (5-5) = 0$

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5}(5-x) = (5-5) = 0$$

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5}(x-5) = 0$$

$$\therefore \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$$

Therefore, f is continuous at x = 5

Case III: c > 5

Then, $f(c) = f(5) = c-5$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(x-5) = c-5$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all real numbers greater than 5.

Hence, f is continuous at every real number and therefore, it is a continuous function.

Question 4:

Prove that the function $f(x) = x^n$ is continuous at x = n, where n is a positive integer.

Answer

The given function is f (x) = x^n

It is evident that f is defined at all positive integers, n, and its value at n is n^n.

Then, $\lim_{x\to n} f(n) = \lim_{x\to n}(x^n) = n^n$

$\therefore \lim_{x\to n} f(x) = f(n)$

Therefore, f is continuous at n, where n is a positive integer.

Question 5:

Is the function f defined by

$$f(x)=\begin{cases}x, & \text{if } x\le 1\\ 5, & \text{if } x>1\end{cases}$$

continuous at x = 0? At x = 1? At x = 2?

Answer

The given function f is $f(x)=\begin{cases}x, & \text{if } x\le 1\\ 5, & \text{if } x>1\end{cases}$

At x = 0,

It is evident that f is defined at 0 and its value at 0 is 0.

Then, $\lim_{x\to 0} f(x) = \lim_{x\to 0} x = 0$

$\therefore \lim_{x\to 0} f(x) = f(0)$

Therefore, f is continuous at x = 0 At x = 1, f is defined at 1 and its value at 1 is 1.

The left hand limit of f at x = 1 is,

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$

The right hand limit of f at x = 1 is,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(5) = 5$$

$$\therefore \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$$

Therefore, f is not continuous at x = 1

At x = 2, f is defined at 2 and its value

at 2 is 5.

$$\text{Then, } \lim_{x\to 2} f(x) = \lim_{x\to 2}(5) = 5$$

$$\therefore \lim_{x\to 2} f(x) = f(2)$$

Therefore, f is continuous at x = 2

Question 6:

Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} 2x+3, & \text{if } x \leq 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$$

Answer

$$f(x) = \begin{cases} 2x+3, & \text{if } x \leq 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$$

It is evident that the given function f is defined at all the points of the real line.

Let c be a point on the real line. Then, three cases arise.

(i) c < 2

(ii) c > 2

(iii) c = 2

Case (i) c < 2

$$\text{Then, } f(c) = 2c + 3$$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(2x + 3) = 2c + 3$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that x < 2

Case (ii) c > 2

$$\text{Then, } f(c) = 2c - 3$$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(2x - 3) = 2c - 3$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that x > 2

Case (iii) c = 2

Then, the left hand limit of f at x = 2 is,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-}(2x + 3) = 2 \times 2 + 3 = 7$$

The right hand limit of f at x = 2 is,

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+}(2x - 3) = 2 \times 2 - 3 = 1$$

It is observed that the left and right hand limit of f at x = 2 do not coincide.

Therefore, f is not continuous at x = 2

Hence, x = 2 is the only point of discontinuity of f.

Question 7:

Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases}|x|+3, \text{if } x\le -3\\ -2x, \text{if } -3<x<3\\ 6x+2, \text{if } x\ge 3\end{cases}$$

Answer

$$f(x)=\begin{cases}|x|+3=-x+3, \text{ if } x\le -3\\ -2x, \text{ if } -3<x<3\\ 6x+2, \text{ if } x\ge 3\end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

$$\text{If } c<-3, \text{ then } f(c)=-c+3$$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(-x+3)=-c+3$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that x < −3

Case II:

$$\text{If } c=-3, \text{ then } f(-3)=-(-3)+3=6$$

$$\lim_{x\to -3^-} f(x)=\lim_{x\to -3^-}(-x+3)=-(-3)+3=6$$

$$\lim_{x\to -3^+} f(x)=\lim_{x\to -3^-}(-2x)=-2\times(-3)=6$$

$$\therefore \lim_{x\to -3} f(x)=f(-3)$$

Therefore, f is continuous at x = −3

Case III:

If $-3 < c < 3$, then $f(c) = -2c$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(-2x) = -2c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous in (−3, 3).

Case IV:

If c = 3, then the left hand limit of f at x = 3 is,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-}(-2x) = -2 \times 3 = -6$$

The right hand limit of f at x = 3 is,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+}(6x + 2) = 6 \times 3 + 2 = 20$$

It is observed that the left and right hand limit of f at x = 3 do not coincide.

Therefore, f is not continuous at x = 3

Case V:

If $c > 3$, then $f(c) = 6c + 2$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(6x + 2) = 6c + 2$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x > 3

Hence, x = 3 is the only point of discontinuity of f.

Question 8:

Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases}\frac{|x|}{x} \text{ if } x \neq 0\\ 0, \text{ if } x=0\end{cases}$$

Answer

$$f(x)=\begin{cases}\frac{|x|}{x} \text{ if } x \neq 0\\ 0, \text{ if } x=0\end{cases}$$

It is known that, $x<0 \Rightarrow |x|=-x$ and $x>0 \Rightarrow |x|=x$

Therefore, the given function can be rewritten as

$$f(x)=\begin{cases}\frac{|x|}{x}=\frac{-x}{x}=-1 \text{ if } x<0\\ 0, \text{ if } x=0\\ \frac{|x|}{x}=\frac{x}{x}=1, \text{ if } x>0\end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c<0$, then $f(c)=-1$

$\lim_{x \to c} f(x)=\lim_{x \to c}(-1)=-1$

$\therefore \lim_{x \to c} f(x)=f(c)$

Therefore, f is continuous at all points x < 0

Case II:

If c = 0, then the left hand limit of f at x = 0 is,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-1) = -1$$

The right hand limit of f at x = 0 is,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(1) = 1$$

It is observed that the left and right hand limit of f at x = 0 do not coincide.

Therefore, f is not continuous at x = 0

Case III:

$$\text{If } c > 0, \text{ then } f(c) = 1$$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(1) = 1$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that x > 0

Hence, x = 0 is the only point of discontinuity of f.

Question 9:

Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases}\frac{x}{|x|}, \text{ if } x<0\\ -1, \text{ if } x\geq 0\end{cases}$$

Answer

$$f(x)=\begin{cases}\frac{x}{|x|}, \text{ if } x<0\\ -1, \text{ if } x\geq 0\end{cases}$$

It is known that, $x<0 \Rightarrow |x|=-x$

Therefore, the given function can be rewritten as

$$f(x)=\begin{cases}\frac{x}{|x|}=\frac{x}{-x}=-1, \text{ if } x<0\\ -1, \text{ if } x\geq 0\end{cases}$$

$$\Rightarrow f(x)=-1 \text{ for all } x\in \mathbf{R}$$

Let c be any real number. Then, $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(-1)=-1$

Also, $f(c)=-1=\lim\limits_{x\to c} f(x)$

Therefore, the given function is a continuous function.

Hence, the given function has no point of discontinuity.

Question 10:

Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases}x+1, & \text{if } x\geq 1\\ x^2+1, & \text{if } x<1\end{cases}$$

Answer

$$f(x)=\begin{cases}x+1, & \text{if } x\geq 1\\ x^2+1, & \text{if } x<1\end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c<1$, then $f(c)=c^2+1$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^2+1\right)=c^2+1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that x < 1

Case II:

If $c=1$, then $f(c)=f(1)=1+1=2$

The left hand limit of f at x = 1 is,

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}\left(x^2+1\right)=1^2+1=2$

The right hand limit of f at x = 1 is,

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(x+1)=1+1=2$

$\therefore \lim\limits_{x\to 1} f(x)=f(1)$

Therefore, f is continuous at x = 1

Case III:

If $c > 1$, then $f(c) = c + 1$

$\lim_{x \to c} f(x) = \lim_{x \to c} (x + 1) = c + 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x > 1

Hence, the given function f has no point of discontinuity.

Question 11:

Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} x^3 - 3, & \text{if } x \le 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$$

Answer

$$f(x) = \begin{cases} x^3 - 3, & \text{if } x \le 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 2$, then $f(c) = c^3 - 3$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (x^3 - 3) = c^3 - 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x < 2

Case II:

$\text{If } c = 2, \text{ then } f(c) = f(2) = 2^3 - 3 = 5$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^3 - 3) = 2^3 - 3 = 5$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + 1) = 2^2 + 1 = 5$

$\therefore \lim_{x \to 2} f(x) = f(2)$

Therefore, f is continuous at x = 2

Case III:

$\text{If } c > 2, \text{ then } f(c) = c^2 + 1$

$\lim_{x \to c} f(x) = \lim_{x \to c} (x^2 + 1) = c^2 + 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x > 2

Thus, the given function f is continuous at every point on the real line.

Hence, f has no point of discontinuity.

Question 12:

Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} x^{10} - 1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1 \end{cases}$$

Answer

$$f(x) = \begin{cases} x^{10} - 1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1 \end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

$$\text{If } c<1, \text{ then } f(c)=c^{10}-1 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}\left(x^{10}-1\right)=c^{10}-1$$
$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that x < 1

Case II:

If c = 1, then the left hand limit of f at x = 1 is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\left(x^{10}-1\right)=1^{10}-1=1-1=0$$

The right hand limit of f at x = 1 is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\left(x^{2}\right)=1^{2}=1$$

It is observed that the left and right hand limit of f at x = 1 do not coincide.

Therefore, f is not continuous at x = 1

Case III:

$$\text{If } c>1, \text{ then } f(c)=c^2$$
$$\lim_{x\to c} f(x)=\lim_{x\to c}\left(x^2\right)=c^2$$
$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that x > 1

Thus, from the above observation, it can be concluded that x = 1 is the only point of discontinuity of f.

Question 13:

Is the function defined by

$$f(x)=\begin{cases} x+5, & \text{if } x \le 1 \\ x-5, & \text{if } x > 1 \end{cases}$$

a continuous function?

Answer

The given function is $f(x)=\begin{cases} x+5, & \text{if } x \le 1 \\ x-5, & \text{if } x > 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c<1$, then $f(c)=c+5$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(x+5)=c+5$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that x < 1

Case II:

If $c=1$, then $f(1)=1+5=6$

The left hand limit of f at x = 1 is,

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(x+5)=1+5=6$

The right hand limit of f at x = 1 is,

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(x-5)=1-5=-4$

It is observed that the left and right hand limit of f at x = 1 do not coincide.

Therefore, f is not continuous at x = 1

Case III:

If $c > 1$, then $f(c) = c - 5$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(x - 5) = c - 5$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x > 1

Thus, from the above observation, it can be concluded that x = 1 is the only point of discontinuity of f.

Question 14:

Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} 3, \text{ if } 0 \le x \le 1 \\ 4, \text{ if } 1 < x < 3 \\ 5, \text{ if } 3 \le x \le 10 \end{cases}$$

Answer

The given function is $f(x) = \begin{cases} 3, \text{ if } 0 \le x \le 1 \\ 4, \text{ if } 1 < x < 3 \\ 5, \text{ if } 3 \le x \le 10 \end{cases}$

The given function is defined at all points of the interval [0, 10].

Let c be a point in the interval [0, 10].

Case I:

$$\text{If } 0 \le c < 1, \text{ then } f(c) = 3 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c}(3) = 3$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous in the interval [0, 1).

Case II:

$\text{If } c = 1, \text{ then } f(3) = 3$

The left hand limit of f at x = 1 is,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(3) = 3$

The right hand limit of f at x = 1 is,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(4) = 4$

It is observed that the left and right hand limits of f at x = 1 do not coincide.

Therefore, f is not continuous at x = 1

Case III:

$$\text{If } 1 < c < 3, \text{ then } f(c) = 4 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c}(4) = 4$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points of the interval (1, 3).

Case IV:

$\text{If } c = 3, \text{ then } f(c) = 5$

The left hand limit of f at x = 3 is,

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-}(4) = 4$

The right hand limit of f at x = 3 is,

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+}(5) = 5$

It is observed that the left and right hand limits of f at x = 3 do not coincide.

Therefore, f is not continuous at x = 3

Case V:

$$\text{If } 3 < c \le 10, \text{ then } f(c) = 5 \text{ and } \lim_{x\to c} f(x) = \lim_{x\to c}(5) = 5$$

$$\lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points of the interval (3, 10].

Hence, f is not continuous at x = 1 and x = 3

Question 15:

Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

Answer

The given function is $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

$$\text{If } c < 0, \text{ then } f(c) = 2c$$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(2x) = 2c$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that x < 0

Case II:

$$\text{If } c=0, \text{ then } f(c)=f(0)=0$$

The left hand limit of f at x = 0 is,

$$\lim_{x\to0^-} f(x)=\lim_{x\to0^-}(2x)=2\times0=0$$

The right hand limit of f at x = 0 is,

$$\lim_{x\to0^+} f(x)=\lim_{x\to0^+}(0)=0$$

$$\therefore \lim_{x\to0} f(x)=f(0)$$

Therefore, f is continuous at x = 0

Case III:

$$\text{If } 0<c<1, \text{ then } f(x)=0 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(0)=0$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points of the interval (0, 1).

Case IV:

$$\text{If } c=1, \text{ then } f(c)=f(1)=0$$

The left hand limit of f at x = 1 is,

$$\lim_{x\to1^-} f(x)=\lim_{x\to1^-}(0)=0$$

The right hand limit of f at x = 1 is,

$$\lim_{x\to1^+} f(x)=\lim_{x\to1^+}(4x)=4\times1=4$$

It is observed that the left and right hand limits of f at x = 1 do not coincide.

Therefore, f is not continuous at x = 1

Case V:

$$\text{If } c < 1, \text{ then } f(c) = 4c \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (4x) = 4c$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that x > 1

Hence, f is not continuous only at x = 1

Question 16:

Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} -2, \text{ if } x \le -1 \\ 2x, \text{ if } -1 < x \le 1 \\ 2, \text{ if } x > 1 \end{cases}$$

Answer

$$f(x) = \begin{cases} -2, \text{ if } x \le -1 \\ 2x, \text{ if } -1 < x \le 1 \\ 2, \text{ if } x > 1 \end{cases}$$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

$$\text{If } c < -1, \text{ then } f(c) = -2 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (-2) = -2$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that x < −1

Case II:

If $c=-1$, then $f(c)=f(-1)=-2$

The left hand limit of f at x = −1 is,

$$\lim_{x\to -1^-} f(x)=\lim_{x\to -1^-}(-2)=-2$$

The right hand limit of f at x = −1 is,

$$\lim_{x\to -1^+} f(x)=\lim_{x\to -1^+}(2x)=2\times(-1)=-2$$

$$\therefore \lim_{x\to -1} f(x)=f(-1)$$

Therefore, f is continuous at x = −1

Case III:

$$\text{If } -1<c<1, \text{ then } f(c)=2c$$
$$\lim_{x\to c} f(x)=\lim_{x\to c}(2x)=2c$$
$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points of the interval (−1, 1).

Case IV:

If $c=1$, then $f(c)=f(1)=2\times 1=2$

The left hand limit of f at x = 1 is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(2x)=2\times 1=2$$

The right hand limit of f at x = 1 is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} 2=2$$

$$\therefore \lim_{x\to 1} f(x)=f(c)$$

Therefore, f is continuous at x = 2

Case V:

If $c > 1$, then $f(c) = 2$ and $\lim_{x\to c} f(x) = \lim_{x\to c}(2) = 2$

$\lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x > 1

Thus, from the above observations, it can be concluded that f is continuous at all points of the real line.

Question 17:

Find the relationship between a and b so that the function f defined by

$$f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$$

is continuous at x = 3.

Answer

$$f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$$

If f is continuous at x = 3, then

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) \qquad \ldots(1)$$

Also,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax+1) = 3a+1$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (bx+3) = 3b+3$$

$$f(3) = 3a+1$$

Therefore, from (1), we obtain

$$3a+1 = 3b+3 = 3a+1$$

$$\Rightarrow 3a+1 = 3b+3$$

$$\Rightarrow 3a = 3b+2$$

$$\Rightarrow a = b+\frac{2}{3}$$

Therefore, the required relationship is given by, $a = b+\frac{2}{3}$

Question 18:

For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$$

continuous at x = 0? What about continuity at x = 1?

Answer

The given function is $f(x)=\begin{cases}\lambda(x^2-2x), & \text{if } x\le 0\\ 4x+1, & \text{if } x>0\end{cases}$

If f is continuous at x = 0, then

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$
$$\Rightarrow \lim_{x\to 0^-} \lambda(x^2-2x)=\lim_{x\to 0^+}(4x+1)=\lambda(0^2-2\times 0)$$
$$\Rightarrow \lambda(0^2-2\times 0)=4\times 0+1=0$$
$$\Rightarrow 0=1=0, \text{ which is not possible}$$

Therefore, there is no value of λ for which f is continuous at x = 0 At x = 1,

f (1) = 4x + 1 = 4 × 1 + 1 = 5

$$\lim_{x\to 1}(4x+1)=4\times 1+1=5$$
$$\therefore \lim_{x\to 1} f(x)=f(1)$$

Therefore, for any values of λ, f is continuous at x = 1

Question 19:

Show that the function defined by $g(x)=x-[x]$ is discontinuous at all integral point.

Here $[x]$ denotes the greatest integer less than or equal to x.

Answer

The given function is $g(x)=x-[x]$

It is evident that g is defined at all integral points.

Let n be an integer.

Then,

$g(n) = n - [n] = n - n = 0$

The left hand limit of f at x = n is,

$\lim_{x\to n^-} g(x) = \lim_{x\to n^-} (x - [x]) = \lim_{x\to n^-} (x) - \lim_{x\to n^-} [x] = n - (n-1) = 1$

The right hand limit of f at x = n is,

$\lim_{x\to n^+} g(x) = \lim_{x\to n^+} (x - [x]) = \lim_{x\to n^+} (x) - \lim_{x\to n^+} [x] = n - n = 0$

It is observed that the left and right hand limits of f at x = n do not coincide.

Therefore, f is not continuous at x = n

Hence, g is discontinuous at all integral points.

Question 20:

Is the function $f(x) = x^2 - \sin x + 5$ defined by continuous at x = p?

Answer

The given function is $f(x) = x^2 - \sin x + 5$

It is evident that f is defined at x = p

At $x = \pi,\ f(x) = f(\pi) = \pi^2 - \sin\pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$

Consider $\lim\limits_{x\to\pi} f(x) = \lim\limits_{x\to\pi}\left(x^2 - \sin x + 5\right)$

Put $x = \pi + h$

If $x \to \pi$, *then it is evident that* $h \to 0$

$$\begin{aligned}
\therefore \lim_{x\to\pi} f(x) &= \lim_{x\to\pi}\left(x^2 - \sin x + 5\right)\\
&= \lim_{h\to 0}\left[(\pi+h)^2 - \sin(\pi+h) + 5\right]\\
&= \lim_{h\to 0}(\pi+h)^2 - \lim_{h\to 0}\sin(\pi+h) + \lim_{h\to 0} 5\\
&= (\pi+0)^2 - \lim_{h\to 0}\left[\sin\pi\cosh + \cos\pi\sinh\right] + 5\\
&= \pi^2 - \lim_{h\to 0}\sin\pi\cosh - \lim_{h\to 0}\cos\pi\sinh + 5\\
&= \pi^2 - \sin\pi\cos 0 - \cos\pi\sin 0 + 5\\
&= \pi^2 - 0\times 1 - (-1)\times 0 + 5\\
&= \pi^2 + 5
\end{aligned}$$

$\therefore \lim\limits_{x\to\pi} f(x) = f(\pi)$

Therefore, the given function f is continuous at x = π

Question 21:

Discuss the continuity of the following functions.

(a) f (x) = sin x + cos x

(b) f (x) = sin x − cos x

(c) f (x) = sin x × cos x

Answer

It is known that if g and h are two continuous functions, then

$g+h,\ g-h,$ and $g.h$ are also continuous.

It has to proved first that g (x) = sin x and h (x) = cos x are continuous functions.

Let g (x) = sin x

It is evident that g (x) = sin x is defined for every real number.

Let c be a real number. Put x = c + h

If x → c, then h → 0

$$g(c)=\sin c$$
$$\lim_{x\to c} g(x)=\lim_{x\to c}\sin x$$
$$=\lim_{h\to 0}\sin(c+h)$$
$$=\lim_{h\to 0}[\sin c\cos h+\cos c\sin h]$$
$$=\lim_{h\to 0}(\sin c\cos h)+\lim_{h\to 0}(\cos c\sin h)$$
$$=\sin c\cos 0+\cos c\sin 0$$
$$=\sin c+0$$
$$=\sin c$$
$$\therefore\ \lim_{x\to c} g(x)=g(c)$$

Therefore, g is a continuous function.

Let h (x) = cos x

It is evident that h (x) = cos x is defined for every real number.

Let c be a real number. Put x = c + h

If x → c, then h → 0 h (c) = cos c

$$\lim_{x \to c} h(x) = \lim_{x \to c} \cos x$$
$$= \lim_{h \to 0} \cos(c+h)$$
$$= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h]$$
$$= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h$$
$$= \cos c \cos 0 - \sin c \sin 0$$
$$= \cos c \times 1 - \sin c \times 0$$
$$= \cos c$$
$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, h is a continuous function.

Therefore, it can be concluded that

(a) f (x) = g (x) + h (x) = sin x + cos x is a continuous function

(b) f (x) = g (x) − h (x) = sin x − cos x is a continuous function

(c) f (x) = g (x) × h (x) = sin x × cos x is a continuous function Question 22:

Discuss the continuity of the cosine, cosecant, secant and cotangent functions,

Answer

It is known that if g and h are two continuous functions, then

$(i)\ \frac{h(x)}{g(x)},\ g(x) \neq 0$ is continuous

$(ii)\ \frac{1}{g(x)},\ g(x) \neq 0$ is continuous

$(iii)\ \frac{1}{h(x)},\ h(x) \neq 0$ is continuous

It has to be proved first that g (x) = sin x and h (x) = cos x are continuous functions.

Let g (x) = sin x

It is evident that g (x) = sin x is defined for every real number.

Let c be a real number. Put x = c + h

If x $\rightarrow$ c, then h $\rightarrow$ 0

$g(c) = \sin c$

$$\lim_{x \to c} g(x) = \lim_{x \to c} \sin x$$
$$= \lim_{h \to 0} \sin(c+h)$$
$$= \lim_{h \to 0} [\sin c \cos h + \cos c \sin h]$$
$$= \lim_{h \to 0} (\sin c \cos h) + \lim_{h \to 0} (\cos c \sin h)$$
$$= \sin c \cos 0 + \cos c \sin 0$$
$$= \sin c + 0$$
$$= \sin c$$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, g is a continuous function.

Let h (x) = cos x

It is evident that h (x) = cos x is defined for every real number.

Let c be a real number. Put x = c + h

If x $\rightarrow$ c, then h $\rightarrow$ 0

h (c) = cos c

$$\lim_{x\to c} h(x) = \lim_{x\to c} \cos x$$
$$= \lim_{h\to 0} \cos(c+h)$$
$$= \lim_{h\to 0} [\cos c \cos h - \sin c \sin h]$$
$$= \lim_{h\to 0} \cos c \cos h - \lim_{h\to 0} \sin c \sin h$$
$$= \cos c \cos 0 - \sin c \sin 0$$
$$= \cos c \times 1 - \sin c \times 0$$
$$= \cos c$$
$$\therefore \lim_{x\to c} h(x) = h(c)$$

Therefore, h (x) = cos x is continuous function.

It can be concluded that,

$$\operatorname{cosec} x = \frac{1}{\sin x},\ \sin x \neq 0 \text{ is continuous}$$
$$\Rightarrow \operatorname{cosec} x,\ x \neq n\pi\ (n \in Z) \text{ is continuous}$$

Therefore, cosecant is continuous except at x = np, n Î Z

$$\sec x = \frac{1}{\cos x},\ \cos x \neq 0 \text{ is continuous}$$
$$\Rightarrow \sec x,\ x \neq (2n+1)\frac{\pi}{2}\ (n \in \mathbf{Z}) \text{ is continuous}$$

Therefore, secant is continuous except at $x = (2n+1)\frac{\pi}{2}\ (n \in \mathbf{Z})$

$$\cot x = \frac{\cos x}{\sin x},\ \sin x \neq 0 \text{ is continuous}$$
$$\Rightarrow \cot x,\ x \neq n\pi\ (n \in Z) \text{ is continuous}$$

Therefore, cotangent is continuous except at x = np, n Î Z

Question 23:

Find the points of discontinuity of f, where

$$f(x)=\begin{cases}\frac{\sin x}{x}, & \text{if } x<0\\ x+1, & \text{if } x\geq 0\end{cases}$$

Answer

$$f(x)=\begin{cases}\frac{\sin x}{x}, & \text{if } x<0\\ x+1, & \text{if } x\geq 0\end{cases}$$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

$$\text{If } c<0, \text{ then } f(c)=\frac{\sin c}{c} \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}\left(\frac{\sin x}{x}\right)=\frac{\sin c}{c}$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that x < 0

Case II:

$$\text{If } c>0, \text{ then } f(c)=c+1 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(x+1)=c+1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that x > 0

Case III:

If $c=0$, then $f(c)=f(0)=0+1=1$

The left hand limit of f at x = 0 is,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}\frac{\sin x}{x}=1$$

The right hand limit of f at x = 0 is,

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(x+1)=1$$

$$\therefore \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$

Therefore, f is continuous at x = 0

From the above observations, it can be concluded that f is continuous at all points of the real line.

Thus, f has no point of discontinuity.

Question 24:

Determine if f defined by

$$f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$$

is a continuous function?

Answer

$$f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$$ It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

$$\text{If } c\neq 0,\ \text{then } f(c)=c^2\sin\frac{1}{c}$$

$$\lim_{x\to c} f(x)=\lim_{x\to c}\left(x^2\sin\frac{1}{x}\right)=\left(\lim_{x\to c} x^2\right)\left(\lim_{x\to c}\sin\frac{1}{x}\right)=c^2\sin\frac{1}{c}$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x ≠ 0

Case II:

$$\text{If } c=0,\ \text{then } f(0)=0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right)$$

It is known that, $-1 \le \sin\frac{1}{x} \le 1,\ x \ne 0$

$$\Rightarrow -x^2 \le \sin\frac{1}{x} \le x^2$$

$$\Rightarrow \lim_{x\to 0}\left(-x^2\right) \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le \lim_{x\to 0} x^2$$

$$\Rightarrow 0 \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le 0$$

$$\Rightarrow \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$$\therefore \lim_{x\to 0^-} f(x) = 0$$

Similarly, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$

$$\therefore \lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Question 25:

Examine the continuity of f, where f is defined by Answer

$$f(x)=\begin{cases}\sin x-\cos x, \text{if } x\neq 0\\ -1 \qquad \text{if } x=0\end{cases}$$

$$f(x)=\begin{cases}\sin x-\cos x, \text{if } x\neq 0\\ -1 \qquad \text{if } x=0\end{cases}$$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

$$\text{If } c\neq 0, \text{ then } f(c)=\sin c-\cos c$$
$$\lim_{x\to c} f(x)=\lim_{x\to c}(\sin x-\cos x)=\sin c-\cos c$$
$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that x ≠ 0

Case II:

$$\text{If } c=0, \text{ then } f(0)=-1$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}(\sin x-\cos x)=\sin 0-\cos 0=0-1=-1$$
$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0}(\sin x-\cos x)=\sin 0-\cos 0=0-1=-1$$
$$\therefore \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$

Therefore, f is continuous at x = 0

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Question 26:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq\dfrac{\pi}{2}\\ 3, & \text{if } x=\dfrac{\pi}{2}\end{cases} \quad \text{at } x=\frac{\pi}{2}$$

Answer

$$f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq\dfrac{\pi}{2}\\ 3, & \text{if } x=\dfrac{\pi}{2}\end{cases}$$

The given function f is continuous at $x=\frac{\pi}{2}$, if f is defined at and if the value of the f at $x=\frac{\pi}{2}$ equals the limit of f at $x=\frac{\pi}{2}$.

$x=\frac{\pi}{2}$

It is evident that f is defined at $x=\frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right)=3$

$$\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}$$

Put $x=\frac{\pi}{2}+h$

Then, $x\to\frac{\pi}{2}\Rightarrow h\to 0$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$$

$$=k\lim_{h\to 0}\frac{-\sin h}{-2h}=\frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h}=\frac{k}{2}.1=\frac{k}{2}$$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)=f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{k}{2}=3$$

$$\Rightarrow k=6$$

Therefore, the required value of k is 6.

Question 27:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x)=\begin{cases}kx^2, & \text{if } x\le 2\\ 3, & \text{if } x>2\end{cases} \qquad \text{at } x=2$$

Answer

The given function is $f(x)=\begin{cases}kx^2, & \text{if } x\le 2\\ 3, & \text{if } x>2\end{cases}$

The given function f is continuous at x = 2, if f is defined at x = 2 and if the value of f at x = 2 equals the limit of f at x = 2

It is evident that f is defined at x = 2 and $f(2) = k(2)^2 = 4k$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$
$$\Rightarrow \lim_{x \to 2^-} (kx^2) = \lim_{x \to 2^+} (3) = 4k$$
$$\Rightarrow k \times 2^2 = 3 = 4k$$
$$\Rightarrow 4k = 3 = 4k$$
$$\Rightarrow 4k = 3$$
$$\Rightarrow k = \frac{3}{4}$$

Therefore, the required value of k is $\frac{3}{4}$.

Question 28:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} kx + 1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases} \quad \text{at } x = \pi$$

Answer

The given function is $f(x) = \begin{cases} kx + 1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$

The given function f is continuous at x = p, if f is defined at x = p and if the value of f at x = p equals the limit of f at x = p

It is evident that f is defined at x = p and $f(\pi) = k\pi + 1$

$$\lim_{x\to \pi^-} f(x) = \lim_{x\to\pi^+} f(x) = f(\pi)$$

$$\Rightarrow \lim_{x\to \pi^-} (kx+1) = \lim_{x\to\pi^+} \cos x = k\pi + 1$$

$$\Rightarrow k\pi + 1 = \cos\pi = k\pi + 1$$

$$\Rightarrow k\pi + 1 = -1 = k\pi + 1$$

$$\Rightarrow k = -\frac{2}{\pi}$$

Therefore, the required value of k is $-\frac{2}{\pi}$.

Question 29:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} kx+1, & \text{if } x \le 5 \\ 3x-5, & \text{if } x > 5 \end{cases} \qquad \text{at } x = 5$$

Answer

$$f(x) = \begin{cases} kx+1, & \text{if } x \le 5 \\ 3x-5, & \text{if } x > 5 \end{cases}$$

The given function f is continuous at x = 5, if f is defined at x = 5 and if the value of f at x = 5 equals the limit of f at x = 5

It is evident that f is defined at x = 5 and $f(5) = kx + 1 = 5k + 1$

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = f(5)$$

$$\Rightarrow \lim_{x \to 5^-} (kx + 1) = \lim_{x \to 5^+} (3x - 5) = 5k + 1$$

$$\Rightarrow 5k + 1 = 15 - 5 = 5k + 1$$

$$\Rightarrow 5k + 1 = 10$$

$$\Rightarrow 5k = 9$$

$$\Rightarrow k = \frac{9}{5}$$

Therefore, the required value of k is $\frac{9}{5}$.

Question 30:

Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$$

is a continuous function.

Answer

$$f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$$

It is evident that the given function f is defined at all points of the real line.

If f is a continuous function, then f is continuous at all real numbers.

In particular, f is continuous at x = 2 and x = 10

Since f is continuous at x = 2, we obtain

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$
$$\Rightarrow \lim_{x\to 2^-} (5) = \lim_{x\to 2^+} (ax+b) = 5$$
$$\Rightarrow 5 = 2a + b = 5$$
$$\Rightarrow 2a + b = 5 \qquad ...(1)$$

Since f is continuous at x = 10, we obtain

$$\lim_{x\to 10^-} f(x) = \lim_{x\to 10^+} f(x) = f(10)$$
$$\Rightarrow \lim_{x\to 10^-} (ax+b) = \lim_{x\to 10^+} (21) = 21$$
$$\Rightarrow 10a + b = 21 = 21$$
$$\Rightarrow 10a + b = 21 \qquad ...(2)$$

On subtracting equation (1) from equation (2), we obtain

8a = 16

⇒ a = 2

By putting a = 2 in equation (1), we obtain

2 × 2 + b = 5

⇒ 4 + b = 5
⇒ b = 1

Therefore, the values of a and b for which f is a continuous function are 2 and 1 respectively.

Question 31:

Show that the function defined by f (x) = cos (x^2) is a continuous function.

Answer

The given function is f (x) = cos (x^2)

This function f is defined for every real number and f can be written as the composition of two functions as,

f = g o h, where g (x) = cos x and h (x) = x^2

$$\left[\because (goh)(x) = g\left(h(x)\right) = g\left(x^2\right) = \cos\left(x^2\right) = f(x)\right]$$

It has to be first proved that g (x) = cos x and h (x) = x^2 are continuous functions.

It is evident that g is defined for every real number.

Let c be a real number.

Then, g (c) = cos c

$$\text{Put } x = c + h$$
$$\text{If } x \to c, \text{ then } h \to 0$$
$$\begin{aligned}\lim_{x\to c} g(x) &= \lim_{x\to c} \cos x \\ &= \lim_{h\to 0} \cos(c+h) \\ &= \lim_{h\to 0}\left[\cos c \cos h - \sin c \sin h\right] \\ &= \lim_{h\to 0} \cos c \cos h - \lim_{h\to 0} \sin c \sin h \\ &= \cos c \cos 0 - \sin c \sin 0 \\ &= \cos c \times 1 - \sin c \times 0 \\ &= \cos c\end{aligned}$$
$$\therefore \lim_{x\to c} g(x) = g(c)$$

Therefore, g (x) = cos x is continuous function.
h (x) = x^2

Clearly, h is defined for every real number.

Let k be a real number, then h (k) = k^2

$$\lim_{x\to k} h(x) = \lim_{x\to k} x^2 = k^2$$
$$\therefore \lim_{x\to k} h(x) = h(k)$$

Therefore, h is a continuous function.

It is known that for real valued functions g and h,such that (g o h) is defined at c, if g is continuous at c and if f is continuous at g (c), then (f o g) is continuous at c.

Therefore, $f(x) = (goh)(x) = \cos\left(x^2\right)$ is a continuous function.

Question 32:

Show that the function defined by $f(x)=|\cos x|$ is a continuous function.

Answer

The given function is $f(x)=|\cos x|$

This function f is defined for every real number and f can be written as the composition of two functions as,

f = g o h, where $g(x)=|x|$ and $h(x)=\cos x$

$$\left[\because (goh)(x)=g(h(x))=g(\cos x)=|\cos x|=f(x)\right]$$

It has to be first proved that $g(x)=|x|$ and $h(x)=\cos x$ are continuous functions.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\geq 0\end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

$$\text{If } c<0, \text{ then } g(c)=-c \text{ and } \lim_{x\to c} g(x)=\lim_{x\to c}(-x)=-c$$
$$\therefore \lim_{x\to c} g(x)=g(c)$$

Therefore, g is continuous at all points x, such that x < 0

Case II:

$$\text{If } c>0, \text{ then } g(c)=c \text{ and } \lim_{x\to c} g(x)=\lim_{x\to c} x=c$$
$$\therefore \lim_{x\to c} g(x)=g(c)$$

Therefore, g is continuous at all points x, such that x > 0

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} (-x) = 0$$
$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (x) = 0$$
$$\therefore \lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} (x) = g(0)$$

Therefore, g is continuous at x = 0

From the above three observations, it can be concluded that g is continuous at all points.

h (x) = cos x

It is evident that h (x) = cos x is defined for every real number.

Let c be a real number. Put x = c + h

If x → c, then h → 0 h (c) = cos c

$$\begin{aligned}\lim_{x\to c} h(x) &= \lim_{x\to c} \cos x \\ &= \lim_{h\to 0} \cos(c+h) \\ &= \lim_{h\to 0} [\cos c \cos h - \sin c \sin h] \\ &= \lim_{h\to 0} \cos c \cos h - \lim_{h\to 0} \sin c \sin h \\ &= \cos c \cos 0 - \sin c \sin 0 \\ &= \cos c \times 1 - \sin c \times 0 \\ &= \cos c\end{aligned}$$

$$\therefore \lim_{x\to c} h(x) = h(c)$$

Therefore, h (x) = cos x is a continuous function.

It is known that for real valued functions g and h,such that (g o h) is defined at c, if g is continuous at c and if f is continuous at g (c), then (f o g) is continuous at c.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\cos x) = |\cos x|$ is a continuous function.

Question 33:

Examine that $\sin|x|$

Answer

$\text{Let } f(x)=\sin|x|$ is a continuous function.

This function f is defined for every real number and f can be written as the composition of two functions as,

f = g o h, where $g(x)=|x| \text{ and } h(x)=\sin x$

$$\left[\because (goh)(x)=g(h(x))=g(\sin x)=|\sin x|=f(x)\right]$$

It has to be proved first that $g(x)=|x| \text{ and } h(x)=\sin x$ are continuous functions.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\geq 0\end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

$$\text{If } c<0, \text{ then } g(c)=-c \text{ and } \lim_{x\to c} g(x)=\lim_{x\to c}(-x)=-c$$
$$\therefore \lim_{x\to c} g(x)=g(c)$$

Therefore, g is continuous at all points x, such that x < 0

Case II:

$$\text{If } c > 0, \text{ then } g(c) = c \text{ and } \lim_{x \to c} g(x) = \lim_{x \to c} x = c$$
$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, g is continuous at all points x, such that x > 0

Case III:

$$\text{If } c = 0, \text{ then } g(c) = g(0) = 0$$
$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$$
$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$$
$$\therefore \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} (x) = g(0)$$

Therefore, g is continuous at x = 0

From the above three observations, it can be concluded that g is continuous at all points.

h (x) = sin x

It is evident that h (x) = sin x is defined for every real number.

Let c be a real number. Put x = c + k

If x → c, then k → 0 h (c) = sin c

$$h(c) = \sin c$$
$$\lim_{x \to c} h(x) = \lim_{x \to c} \sin x$$
$$= \lim_{k \to 0} \sin(c + k)$$
$$= \lim_{k \to 0} [\sin c \cos k + \cos c \sin k]$$
$$= \lim_{k \to 0} (\sin c \cos k) + \lim_{h \to 0} (\cos c \sin k)$$
$$= \sin c \cos 0 + \cos c \sin 0$$
$$= \sin c + 0$$
$$= \sin c$$
$$\therefore \lim_{x \to c} h(x) = g(c)$$

Therefore, h is a continuous function.

It is known that for real valued functions g and h,such that (g o h) is defined at c, if g is continuous at c and if f is continuous at g (c), then (f o g) is continuous at c.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\sin x) = |\sin x|$ is a continuous function.

Question 34:

Find all the points of discontinuity of f defined by $f(x) = |x| - |x+1|$.

Answer

The given function is $f(x) = |x| - |x+1|$

The two functions, g and h, are defined as

$g(x) = |x|$ and $h(x) = |x+1|$

Then, f = g − h

The continuity of g and h is examined first.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\geq 0\end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

$$\text{If } c<0, \text{ then } g(c)=-c \text{ and } \lim_{x\to c} g(x)=\lim_{x\to c}(-x)=-c$$
$$\therefore \lim_{x\to c} g(x)=g(c)$$

Therefore, g is continuous at all points x, such that x < 0

Case II:

$$\text{If } c>0, \text{ then } g(c)=c \text{ and } \lim_{x\to c} g(x)=\lim_{x\to c} x=c$$
$$\therefore \lim_{x\to c} g(x)=g(c)$$

Therefore, g is continuous at all points x, such that x > 0

Case III:

$$\text{If } c=0, \text{ then } g(c)=g(0)=0$$

$$\lim_{x\to 0^-} g(x)=\lim_{x\to 0^-}(-x)=0$$
$$\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+}(x)=0$$
$$\therefore \lim_{x\to 0^-} g(x)=\lim_{x\to 0^+}(x)=g(0)$$

Therefore, g is continuous at x = 0

From the above three observations, it can be concluded that g is continuous at all points.

$h(x)=|x+1|$ can be written as

$$h(x)=\begin{cases}-(x+1), & \text{if, } x<-1\\ x+1, & \text{if } x\geq -1\end{cases}$$

Clearly, h is defined for every real number.
Let c be a real number.

Case I:

$$\text{If } c<-1, \text{ then } h(c)=-(c+1) \text{ and } \lim_{x\to c} h(x)=\lim_{x\to c}\left[-(x+1)\right]=-(c+1)$$

$$\therefore \lim_{x\to c} h(x)=h(c)$$

Therefore, h is continuous at all points x, such that x < −1

Case II:

$$\text{If } c>-1, \text{ then } h(c)=c+1 \text{ and } \lim_{x\to c} h(x)=\lim_{x\to c}(x+1)=c+1$$

$$\therefore \lim_{x\to c} h(x)=h(c)$$

Therefore, h is continuous at all points x, such that x > −1

Case III:

$$\text{If } c=-1, \text{ then } h(c)=h(-1)=-1+1=0$$

$$\lim_{x\to -1^-} h(x)=\lim_{x\to -1^-}\left[-(x+1)\right]=-(-1+1)=0$$

$$\lim_{x\to -1^+} h(x)=\lim_{x\to -1^+}(x+1)=(-1+1)=0$$

$$\therefore \lim_{x\to -1^-} h(x)=\lim_{h\to -1^+} h(x)=h(-1)$$

Therefore, h is continuous at x = −1

From the above three observations, it can be concluded that h is continuous at all points of the real line. g and h are continuous functions. Therefore, f = g − h is also a continuous function.

Therefore, f has no point of discontinuity.

Exercise 5.2

Question 1:

Differentiate the functions with respect to x.

$\sin\left(x^2+5\right)$

Answer

Let $f(x)=\sin\left(x^2+5\right)$, $u(x)=x^2+5$, and $v(t)=\sin t$

Then, $(vou)(x)=v(u(x))=v\left(x^2+5\right)=\tan\left(x^2+5\right)=f(x)$

Thus, f is a composite of two functions.

Put $t = u(x) = x^2 + 5$

Then, we obtain

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(x^2+5)$$

$$\frac{dt}{dx} = \frac{d}{dx}(x^2+5) = \frac{d}{dx}(x^2) + \frac{d}{dx}(5) = 2x + 0 = 2x$$

Therefore, by chain rule, $\frac{df}{dx} = \frac{dv}{dt}\cdot\frac{dt}{dx} = \cos(x^2+5)\times 2x = 2x\cos(x^2+5)$

Alternate method

$$\begin{aligned}\frac{d}{dx}\left[\sin(x^2+5)\right] &= \cos(x^2+5)\cdot\frac{d}{dx}(x^2+5)\\ &= \cos(x^2+5)\cdot\left[\frac{d}{dx}(x^2)+\frac{d}{dx}(5)\right]\\ &= \cos(x^2+5)\cdot[2x+0]\\ &= 2x\cos(x^2+5)\end{aligned}$$

Question 2:

Differentiate the functions with respect to x.

$\cos(\sin x)$

Answer

Let $f(x) = \cos(\sin x), u(x) = \sin x,$ and $v(t) = \cos t$

Then, $(vou)(x) = v(u(x)) = v(\sin x) = \cos(\sin x) = f(x)$

Thus, f is a composite function of two functions.

Put t = u (x) = sin x

$\therefore \frac{dv}{dt} = \frac{d}{dt}[\cos t] = -\sin t = -\sin(\sin x)$

$\frac{dt}{dx} = \frac{d}{dx}(\sin x) = \cos x$

By chain rule, $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x)$

Alternate method

$$\frac{d}{dx}\left[\cos(\sin x)\right] = -\sin(\sin x) \cdot \frac{d}{dx}(\sin x) = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x)$$

Question 3:

Differentiate the functions with respect to x.

$\sin(ax + b)$

Answer

Let $f(x) = \sin(ax+b)$, $u(x) = ax + b$, and $v(t) = \sin t$

Then, $(vou)(x) = v(u(x)) = v(ax+b) = \sin(ax+b) = f(x)$

Thus, f is a composite function of two functions, u and v.

Put t = u (x) = ax + b

Therefore,

$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax+b)$

$\frac{dt}{dx} = \frac{d}{dx}(ax+b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$

Hence, by chain rule, we obtain

$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax+b) \cdot a = a\cos(ax+b)$

Alternate method

$$\frac{d}{dx}\left[\sin(ax+b)\right]=\cos(ax+b).\frac{d}{dx}(ax+b)$$
$$=\cos(ax+b)\cdot\left[\frac{d}{dx}(ax)+\frac{d}{dx}(b)\right]$$
$$=\cos(ax+b).(a+0)$$
$$=a\cos(ax+b)$$

Question 4:

Differentiate the functions with respect to x.

$\sec\left(\tan\left(\sqrt{x}\right)\right)$

Answer

Let $f(x)=\sec\left(\tan\sqrt{x}\right), u(x)=\sqrt{x}, v(t)=\tan t,$ and $w(s)=\sec s$

Then, $(wovou)(x)=w\left[v(u(x))\right]=w\left[v\left(\sqrt{x}\right)\right]=w\left(\tan\sqrt{x}\right)=\sec\left(\tan\sqrt{x}\right)=f(x)$

Thus, f is a composite function of three functions, u, v, and w.

Put $s=v(t)=\tan t$ and $t=u(x)=\sqrt{x}$

Then, $\frac{dw}{ds}=\frac{d}{ds}(\sec s)=\sec s\tan s=\sec(\tan t).\tan(\tan t)$ $[s=\tan t]$

$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)$ $\left[t=\sqrt{x}\right]$

$\frac{ds}{dt}=\frac{d}{dt}(\tan t)=\sec^2 t=\sec^2\sqrt{x}$

$\frac{dt}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}\cdot x^{\frac{1}{2}-1}=\frac{1}{2\sqrt{x}}$

Hence, by chain rule, we obtain

$$\frac{dt}{dx}=\frac{dw}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\times\sec^2\sqrt{x}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)$$

$$=\frac{\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\sec\left(\tan\sqrt{x}\right)\right]=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right)\cdot\frac{d}{dx}\left(\tan\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}$$

$$=\frac{\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Question 5:

Differentiate the functions with respect to x.

$$\frac{\sin(ax+b)}{\cos(cx+d)}$$

Answer

The given function is $f(x)=\frac{\sin(ax+b)}{\cos(cx+d)}=\frac{g(x)}{h(x)}$, where g (x) = sin (ax + b) and h (x) = cos (cx + d)

$\therefore f'=\frac{g'h-gh'}{h^2}$

Consider $g(x)=\sin(ax+b)$

Let $u(x)=ax+b, v(t)=\sin t$

Then, $(vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=g(x)$

∴ g is a composite function of two functions, u and v.

Put $t=u(x)=ax+b$

$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b)$

$\frac{dt}{dx}=\frac{d}{dx}(ax+b)=\frac{d}{dx}(ax)+\frac{d}{dx}(b)=a+0=a$

Therefore, by chain rule, we obtain

$g'=\frac{dg}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=\cos(ax+b)\cdot a=a\cos(ax+b)$

Consider $h(x)=\cos(cx+d)$

Let $p(x)=cx+d,\ q(y)=\cos y$

Then, $(qop)(x)=q(p(x))=q(cx+d)=\cos(cx+d)=h(x)$

∴h is a composite function of two functions, p and q.

Put y = p (x) = cx + d

$$\frac{dq}{dy}=\frac{d}{dy}(\cos y)=-\sin y=-\sin(cx+d)$$

$$\frac{dy}{dx}=\frac{d}{dx}(cx+d)=\frac{d}{dx}(cx)+\frac{d}{dx}(d)=c$$

Therefore, by chain rule, we obtain

$$h'=\frac{dh}{dx}=\frac{dq}{dy}\cdot\frac{dy}{dx}=-\sin(cx+d)\times c=-c\sin(cx+d)$$

$$\therefore f'=\frac{a\cos(ax+b)\cdot\cos(cx+d)-\sin(ax+b)\{-c\sin(cx+d)\}}{[\cos(cx+d)]^2}$$

$$=\frac{a\cos(ax+b)}{\cos(cx+d)}+c\sin(ax+b)\cdot\frac{\sin(cx+d)}{\cos(cx+d)}\times\frac{1}{\cos(cx+d)}$$

$$=a\cos(ax+b)\sec(cx+d)+c\sin(ax+b)\tan(cx+d)\sec(cx+d)$$

Question 6:

Differentiate the functions with respect to x.

$$\cos x^3.\sin^2(x^5)$$

Answer

The given function is $\cos x^3 . \sin^2\left(x^5\right)$

$$\frac{d}{dx}\left[\cos x^3 \cdot \sin^2\left(x^5\right)\right] = \sin^2\left(x^5\right) \times \frac{d}{dx}\left(\cos x^3\right) + \cos x^3 \times \frac{d}{dx}\left[\sin^2\left(x^5\right)\right]$$

$$= \sin^2\left(x^5\right) \times \left(-\sin x^3\right) \times \frac{d}{dx}\left(x^3\right) + \cos x^3 \times 2\sin\left(x^5\right) \cdot \frac{d}{dx}\left[\sin x^5\right]$$

$$= -\sin x^3 \sin^2\left(x^5\right) \times 3x^2 + 2\sin x^5 \cos x^3 \cdot \cos x^5 \times \frac{d}{dx}\left(x^5\right)$$

$$= -3x^2 \sin x^3 \cdot \sin^2\left(x^5\right) + 2\sin x^5 \cos x^5 \cos x^3 \cdot \times 5x^4$$

$$= 10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2\left(x^5\right)$$

Question 7:

Differentiate the functions with respect to x.

$$2\sqrt{\cot\left(x^2\right)}$$

Answer

$$\frac{d}{dx}\left[2\sqrt{\cot\left(x^2\right)}\right]$$

$$= 2\cdot\frac{1}{2\sqrt{\cot\left(x^2\right)}}\times\frac{d}{dx}\left[\cot\left(x^2\right)\right]$$

$$= \sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times -\operatorname{cosec}^2\left(x^2\right)\times\frac{d}{dx}\left(x^2\right)$$

$$= -\sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times\frac{1}{\sin^2\left(x^2\right)}\times(2x)$$

$$= \frac{-2x}{\sqrt{\cos x^2}\sqrt{\sin x^2}\sin x^2}$$

$$= \frac{-2\sqrt{2}\,x}{\sqrt{2\sin x^2\cos x^2}\sin x^2}$$

$$= \frac{-2\sqrt{2}\,x}{\sin x^2\sqrt{\sin 2x^2}}$$

Question 8:

Differentiate the functions with respect to x.

$\cos\left(\sqrt{x}\right)$

Answer

Let $f(x) = \cos\left(\sqrt{x}\right)$

Also, let $u(x) = \sqrt{x}$

And, $v(t) = \cos t$

Then, $(vou)(x) = v\left(u(x)\right)$

$$= v\left(\sqrt{x}\right)$$
$$= \cos\sqrt{x}$$
$$= f(x)$$

Clearly, f is a composite function of two functions, u and v, such that

$t = u(x) = \sqrt{x}$

Then, $\frac{dt}{dx} = \frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$

And, $\frac{dv}{dt} = \frac{d}{dt}(\cos t) = -\sin t$

$= -\sin\left(\sqrt{x}\right)$

By using chain rule, we obtain

$$\frac{dt}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

$$= -\sin\left(\sqrt{x}\right) \cdot \frac{1}{2\sqrt{x}}$$

$$= -\frac{1}{2\sqrt{x}} \sin\left(\sqrt{x}\right)$$

$$= -\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\cos\left(\sqrt{x}\right)\right] = -\sin\left(\sqrt{x}\right) \cdot \frac{d}{dx}\left(\sqrt{x}\right)$$

$$= -\sin\left(\sqrt{x}\right) \times \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= -\sin\sqrt{x} \times \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Question 9:

Prove that the function f given by

$f(x)=|x-1|,\ x\in\mathbf{R}$ is notdifferentiable at x = 1.

Answer

The given function is $f(x)=|x-1|,\ x\in\mathbf{R}$

It is known that a function f is differentiable at a point x = c in its domain if both

$\lim_{h\to0^-}\frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at x = 1,

consider the left hand limit of f at x = 1

$$\lim_{h\to0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^-}\frac{|1+h-1|-|1-1|}{h}$$

$$=\lim_{h\to0^-}\frac{|h|-0}{h}=\lim_{h\to0^-}\frac{-h}{h}\qquad(h<0\Rightarrow|h|=-h)$$

$$=-1$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^+}\frac{|1+h-1|-|1-1|}{h}$$

$$=\lim_{h\to0^+}\frac{|h|-0}{h}=\lim_{h\to0^+}\frac{h}{h}\qquad(h>0\Rightarrow|h|=h)$$

$$=1$$

Since the left and right hand limits of f at x = 1 are not equal, f is not differentiable at x = 1

Question 10:

Prove that the greatest integer function defined by $f(x)=[x], 0<x<3$ is not differentiable at x = 1 and x = 2.

Answer

The given function f is $f(x)=[x], 0<x<3$

It is known that a function f is differentiable at a point x = c in its domain if both

$\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at x = 1, consider the left hand limit of f at x = 1

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{[1+h]-[1]}{h}$$

$$=\lim_{h\to 0^-}\frac{0-1}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{[1+h]-[1]}{h}$$

$$=\lim_{h\to 0^+}\frac{1-1}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = 1 are not equal, f is not differentiable at x = 1

To check the differentiability of the given function at x = 2, consider the left hand limit of f at x = 2

$$\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^-}\frac{[2+h]-[2]}{h}$$

$$=\lim_{h\to 0^-}\frac{1-2}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^+}\frac{[2+h]-[2]}{h}$$

$$=\lim_{h\to 0^+}\frac{2-2}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = 2 are not equal, f is not differentiable at x = 2

Exercise 5.3

Question 1:

Find $\frac{dy}{dx}$:

$2x+3y=\sin x$

Answer

The given relationship is $2x+3y=\sin x$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(2x+3y)=\frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\cos x$$

$$\Rightarrow 2+3\frac{dy}{dx}=\cos x$$

$$\Rightarrow 3\frac{dy}{dx}=\cos x-2$$

$$\therefore \frac{dy}{dx}=\frac{\cos x-2}{3}$$

Question 2:

Find $\frac{dy}{dx}$

$2x+3y=\sin y$

Answer

The given relationship is $2x+3y=\sin y$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\frac{d}{dx}(\sin y)$$

$$\Rightarrow 2+3\frac{dy}{dx}=\cos y\frac{dy}{dx} \qquad \text{[By using chain rule]}$$

$$\Rightarrow 2=(\cos y-3)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{2}{\cos y-3}$$

Question 3:

Find $\frac{dy}{dx}$

$ax+by^2=\cos y$

Answer

The given relationship is $ax+by^2=\cos y$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(ax)+\frac{d}{dx}(by^2)=\frac{d}{dx}(\cos y)$$

$$\Rightarrow a+b\frac{d}{dx}(y^2)=\frac{d}{dx}(\cos y) \qquad ...(1)$$

Using chain rule, we obtain $\frac{d}{dx}(y^2)=2y\frac{dy}{dx}$ and $\frac{d}{dx}(\cos y)=-\sin y\frac{dy}{dx}$ ...(2)

From (1) and (2), we obtain

$$a+b\times 2y\frac{dy}{dx}=-\sin y\frac{dy}{dx}$$

$$\Rightarrow (2by+\sin y)\frac{dy}{dx}=-a$$

$$\therefore \frac{dy}{dx}=\frac{-a}{2by+\sin y}$$

Question 4:

Find $\frac{dy}{dx}$

$xy + y^2 = \tan x + y$

Answer

The given relationship is $xy + y^2 = \tan x + y$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(xy + y^2\right) = \frac{d}{dx}\left(\tan x + y\right)$$

$$\Rightarrow \frac{d}{dx}(xy) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(\tan x) + \frac{dy}{dx}$$

$$\Rightarrow \left[y \cdot \frac{d}{dx}(x) + x.\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \qquad \text{[Using product rule and chain rule]}$$

$$\Rightarrow y \cdot 1 + x.\frac{dy}{dx} + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{(x + 2y - 1)}$$

Question 5:

Find $\frac{dy}{dx}$

$x^2 + xy + y^2 = 100$

Answer

The given relationship is $x^2 + xy + y^2 = 100$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(x^2 + xy + y^2\right) = \frac{d}{dx}(100)$$

$$\Rightarrow \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(xy) + \frac{d}{dx}\left(y^2\right) = 0$$

$$\Rightarrow 2x + \left[y \cdot \frac{d}{dx}(x) + x \cdot \frac{dy}{dx}\right] + 2y\frac{dy}{dx} = 0 \qquad \text{[Using product rule and chain rule]}$$

$$\Rightarrow 2x + y \cdot 1 + x \cdot \frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2x + y + (x + 2y)\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

Question 6:

Find $\frac{dy}{dx}$

$x^3 + x^2y + xy^2 + y^3 = 81$

Answer

The given relationship is $x^3 + x^2 y + xy^2 + y^3 = 81$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(x^3 + x^2 y + xy^2 + y^3\right) = \frac{d}{dx}(81)$$

$$\Rightarrow \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(x^2 y\right) + \frac{d}{dx}\left(xy^2\right) + \frac{d}{dx}\left(y^3\right) = 0$$

$$\Rightarrow 3x^2 + \left[y\frac{d}{dx}\left(x^2\right) + x^2\frac{dy}{dx}\right] + \left[y^2\frac{d}{dx}(x) + x\frac{d}{dx}\left(y^2\right)\right] + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 + \left[y \cdot 2x + x^2\frac{dy}{dx}\right] + \left[y^2 \cdot 1 + x \cdot 2y \cdot \frac{dy}{dx}\right] + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \left(x^2 + 2xy + 3y^2\right)\frac{dy}{dx} + \left(3x^2 + 2xy + y^2\right) = 0$$

$$\therefore \frac{dy}{dx} = \frac{-\left(3x^2 + 2xy + y^2\right)}{\left(x^2 + 2xy + 3y^2\right)}$$

Question 7:

Find $\frac{dy}{dx}$

$\sin^2 y + \cos xy = \pi$

Answer

The given relationship is $\sin^2 y + \cos xy = \pi$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(\sin^2 y + \cos xy\right) = \frac{d}{dx}(\pi)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 y\right) + \frac{d}{dx}(\cos xy) = 0 \qquad ...(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}\left(\sin^2 y\right)=2\sin y\frac{d}{dx}\left(\sin y\right)=2\sin y\cos y\frac{dy}{dx} \qquad ...(2)$$

$$\frac{d}{dx}\left(\cos xy\right)=-\sin xy\frac{d}{dx}\left(xy\right)=-\sin xy\left[y\frac{d}{dx}(x)+x\frac{dy}{dx}\right]$$

$$=-\sin xy\left[y.1+x\frac{dy}{dx}\right]=-y\sin xy-x\sin xy\frac{dy}{dx} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$2\sin y\cos y\frac{dy}{dx}-y\sin xy-x\sin xy\frac{dy}{dx}=0$$

$$\Rightarrow\left(2\sin y\cos y-x\sin xy\right)\frac{dy}{dx}=y\sin xy$$

$$\Rightarrow\left(\sin 2y-x\sin xy\right)\frac{dy}{dx}=y\sin xy$$

$$\therefore \frac{dy}{dx}=\frac{y\sin xy}{\sin 2y-x\sin xy}$$

Question 8:

Find $\frac{dy}{dx}$

$\sin^2 x+\cos^2 y=1$

Answer

The given relationship is $\sin^2 x+\cos^2 y=1$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(\sin^2 x+\cos^2 y\right)=\frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 x\right)+\frac{d}{dx}\left(\cos^2 y\right)=0$$

$$\Rightarrow 2\sin x\cdot\frac{d}{dx}(\sin x)+2\cos y\cdot\frac{d}{dx}(\cos y)=0$$

$$\Rightarrow 2\sin x\cos x+2\cos y(-\sin y)\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \sin 2x-\sin 2y\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=\frac{\sin 2x}{\sin 2y}$$

Question 9:

Find $\frac{dy}{dx}$

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Answer

The given relationship is $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \sin y = \frac{2x}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y \frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right) \qquad ...(1)$$

The function, $\frac{2x}{1+x^2}$, is of the form of $\frac{u}{v}$.

Therefore, by quotient rule, we obtain

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)=\frac{\left(1+x^2\right).\frac{d}{dx}(2x)-2x\cdot\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$=\frac{\left(1+x^2\right).2-2x\cdot[0+2x]}{\left(1+x^2\right)^2}=\frac{2+2x^2-4x^2}{\left(1+x^2\right)^2}=\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2} \qquad ...(2)$$

Also, $\sin y=\frac{2x}{1+x^2}$

$$\Rightarrow \cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}=\sqrt{\frac{\left(1+x^2\right)^2-4x^2}{\left(1+x^2\right)^2}}$$

$$=\sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}=\frac{1-x^2}{1+x^2} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{1-x^2}{1+x^2}\times\frac{dy}{dx}=\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{1+x^2}$$

Question 10:

Find $\frac{dy}{dx}$

$$y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$$

Answer

The given relationship is $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$$y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow \tan y=\frac{3x-x^3}{1-3x^2} \qquad ...(1)$$

It is known that, $\tan y=\frac{3\tan\frac{y}{3}-\tan^3\frac{y}{3}}{1-3\tan^2\frac{y}{3}}$ $\qquad ...(2)$

Comparing equations (1) and (2), we obtain

$$x=\tan\frac{y}{3}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x)=\frac{d}{dx}\left(\tan\frac{y}{3}\right)$$

$$\Rightarrow 1=\sec^2\frac{y}{3}\cdot\frac{d}{dx}\left(\frac{y}{3}\right)$$

$$\Rightarrow 1=\sec^2\frac{y}{3}\cdot\frac{1}{3}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{3}{\sec^2\frac{y}{3}}=\frac{3}{1+\tan^2\frac{y}{3}}$$

$$\therefore \frac{dy}{dx}=\frac{3}{1+x^2}$$

Question 11:

Find $\frac{dy}{dx}$

$$y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0<x<1$$

Answer

The given relationship is,

$$y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \cos y=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow \frac{1-\tan^2\frac{y}{2}}{1+\tan^2\frac{y}{2}}=\frac{1-x^2}{1+x^2}$$

On comparing L.H.S. and R.H.S. of the above relationship, we obtain

$$\tan\frac{y}{2}=x$$

Differentiating this relationship with respect to x, we obtain

$$\sec^2\frac{y}{2}\cdot\frac{d}{dx}\left(\frac{y}{2}\right)=\frac{d}{dx}(x)$$

$$\Rightarrow \sec^2\frac{y}{2}\times\frac{1}{2}\frac{dy}{dx}=1$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{\sec^2\frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{1+\tan^2\frac{y}{2}}$$

$$\therefore \frac{dy}{dx}=\frac{1}{1+x^2}$$

Question 12:

Find $\frac{dy}{dx}$

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ 0 < x < 1$$

Answer

The given relationship is $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y=\frac{1-x^2}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \quad ...(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}(\sin y)=\cos y\cdot\frac{dy}{dx}$$

$$\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}$$

$$=\sqrt{\frac{\left(1+x^2\right)^2-\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}=\sqrt{\frac{4x^2}{\left(1+x^2\right)^2}}=\frac{2x}{1+x^2}$$

$$\therefore \frac{d}{dx}(\sin y)=\frac{2x}{1+x^2}\frac{dy}{dx} \quad ...(2)$$

$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)=\frac{\left(1+x^2\right)\cdot\left(1-x^2\right)'-\left(1-x^2\right)\cdot\left(1+x^2\right)'}{\left(1+x^2\right)^2} \quad \text{[Using quotient rule]}$$

$$=\frac{\left(1+x^2\right)(-2x)-\left(1-x^2\right)\cdot(2x)}{\left(1+x^2\right)^2}$$

$$=\frac{-2x-2x^3-2x+2x^3}{\left(1+x^2\right)^2}$$

$$=\frac{-4x}{\left(1+x^2\right)^2} \quad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{2x}{1+x^2}\frac{dy}{dx}=\frac{-4x}{\left(1+x^2\right)^2}$$

$$\Rightarrow\frac{dy}{dx}=\frac{-2}{1+x^2}$$

Alternate method

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow\sin y=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow \left(1+x^2\right)\sin y = 1-x^2$$

$$\Rightarrow \left(1+\sin y\right)x^2 = 1-\sin y$$

$$\Rightarrow x^2 = \frac{1-\sin y}{1+\sin y}$$

$$\Rightarrow x^2 = \frac{\left(\cos\frac{y}{2}-\sin\frac{y}{2}\right)^2}{\left(\cos\frac{y}{2}+\sin\frac{y}{2}\right)^2}$$

$$\Rightarrow x = \frac{\cos\frac{y}{2}-\sin\frac{y}{2}}{\cos\frac{y}{2}+\sin\frac{y}{2}}$$

$$\Rightarrow x = \frac{1-\tan\frac{y}{2}}{1+\tan\frac{y}{2}}$$

$$\Rightarrow x = \tan\left(\frac{\pi}{4}-\frac{y}{2}\right)$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x)=\frac{d}{dx}\cdot\left[\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)\right]$$

$$\Rightarrow 1=\sec^2\left(\frac{\pi}{4}-\frac{y}{2}\right)\cdot\frac{d}{dx}\left(\frac{\pi}{4}-\frac{y}{2}\right)$$

$$\Rightarrow 1=\left[1+\tan^2\left(\frac{\pi}{4}-\frac{y}{2}\right)\right]\cdot\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow 1=\left(1+x^2\right)\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^2}$$

Question 13:

Find $\frac{dy}{dx}$

$$y=\cos^{-1}\left(\frac{2x}{1+x^2}\right),-1<x<1$$

Answer

The given relationship is $y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$$
$$\Rightarrow \cos y=\frac{2x}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\cos y)=\frac{d}{dx}\cdot\left(\frac{2x}{1+x^2}\right)$$
$$\Rightarrow -\sin y\cdot\frac{dy}{dx}=\frac{\left(1+x^2\right)\cdot\frac{d}{dx}(2x)-2x\cdot\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$
$$\Rightarrow -\sqrt{1-\cos^2 y}\frac{dy}{dx}=\frac{\left(1+x^2\right)\times 2-2x\cdot 2x}{\left(1+x^2\right)^2}$$
$$\Rightarrow\left[\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}\right]\frac{dy}{dx}=-\left[\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}\right]$$
$$\Rightarrow\sqrt{\frac{\left(1+x^2\right)^2-4x^2}{\left(1+x^2\right)^2}}\frac{dy}{dx}=\frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$
$$\Rightarrow\sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}\frac{dy}{dx}=\frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$
$$\Rightarrow\frac{1-x^2}{1+x^2}\cdot\frac{dy}{dx}=\frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$
$$\Rightarrow\frac{dy}{dx}=\frac{-2}{1+x^2}$$

Question 14:

Find $\frac{dy}{dx}$

$y=\sin^{-1}\left(2x\sqrt{1-x^2}\right),\ -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$

Answer

The given relationship is $y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$$y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$$
$$\Rightarrow \sin y=2x\sqrt{1-x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\cos y\frac{dy}{dx}=2\left[x\frac{d}{dx}\left(\sqrt{1-x^2}\right)+\sqrt{1-x^2}\frac{dx}{dx}\right]$$
$$\Rightarrow \sqrt{1-\sin^2 y}\frac{dy}{dx}=2\left[\frac{x}{2}.\frac{-2x}{\sqrt{1-x^2}}+\sqrt{1-x^2}\right]$$
$$\Rightarrow \sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}\frac{dy}{dx}=2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$$
$$\Rightarrow \sqrt{1-4x^2\left(1-x^2\right)}\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$
$$\Rightarrow \sqrt{\left(1-2x^2\right)^2}\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$
$$\Rightarrow \left(1-2x^2\right)\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$
$$\Rightarrow \frac{dy}{dx}=\frac{2}{\sqrt{1-x^2}}$$

Question 15:

Find $\frac{dy}{dx}$

$y = \sec^{-1}\left(\frac{1}{2x^2 - 1}\right), 0 < x < \frac{1}{\sqrt{2}}$

Answer

The given relationship is $y = \sec^{-1}\left(\frac{1}{2x^2 - 1}\right)$

$$y = \sec^{-1}\left(\frac{1}{2x^2 - 1}\right)$$

$$\Rightarrow \sec y = \frac{1}{2x^2 - 1}$$

$$\Rightarrow \cos y = 2x^2 - 1$$

$$\Rightarrow 2x^2 = 1 + \cos y$$

$$\Rightarrow 2x^2 = 2\cos^2 \frac{y}{2}$$

$$\Rightarrow x = \cos \frac{y}{2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\cos \frac{y}{2}\right)$$

$$\Rightarrow 1 = -\sin \frac{y}{2} \cdot \frac{d}{dx}\left(\frac{y}{2}\right)$$

$$\Rightarrow \frac{-1}{\sin \frac{y}{2}} = \frac{1}{2} \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sin \frac{y}{2}} = \frac{-2}{\sqrt{1 - \cos^2 \frac{y}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1 - x^2}}$$

Exercise 5.4

Question 1:

Differentiate the following w.r.t. x:

$$\frac{e^x}{\sin x}$$

Answer

Let $y = \frac{e^x}{\sin x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\sin x \frac{d}{dx}\left(e^x\right) - e^x \frac{d}{dx}\left(\sin x\right)}{\sin^2 x}$$

$$= \frac{\sin x.\left(e^x\right) - e^x \cdot \left(\cos x\right)}{\sin^2 x}$$

$$= \frac{e^x\left(\sin x - \cos x\right)}{\sin^2 x}, x \neq n\pi, n \in \mathbf{Z}$$

Question 2:

Differentiate the following w.r.t. x:

$e^{\sin^{-1} x}$

Answer

Let $y = e^{\sin^{-1} x}$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin^{-1} x}\right)$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin^{-1} x} \cdot \frac{d}{dx}\left(\sin^{-1} x\right)$$

$$= e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}, x \in (-1,1)$$

Question 2:

Show that the function given by f(x) = e^{2x} is strictly increasing on R.

Answer

Let x_1 and x_2 be any two numbers in R.

Then, we have:

$$x_1 < x_2 \Rightarrow 2x_1 < 2x_2 \Rightarrow e^{2x_1} < e^{2x_2} \Rightarrow f(x_1) < f(x_2)$$

Hence, f is strictly increasing on R.

Question 3:

Differentiate the following w.r.t. x:

e^{x^3}

Answer

Let $y = e^{x^3}$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x^3}\right) = e^{x^3} \cdot \frac{d}{dx}\left(x^3\right) = e^{x^3} \cdot 3x^2 = 3x^2 e^{x^3}$$

Question 4:

Differentiate the following w.r.t. x:

$\sin\left(\tan^{-1} e^{-x}\right)$

Answer

Let $y = \sin\left(\tan^{-1} e^{-x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin\left(\tan^{-1} e^{-x}\right)\right]$$

$$= \cos\left(\tan^{-1} e^{-x}\right) \cdot \frac{d}{dx}\left(\tan^{-1} e^{-x}\right)$$

$$= \cos\left(\tan^{-1} e^{-x}\right) \cdot \frac{1}{1+\left(e^{-x}\right)^2} \cdot \frac{d}{dx}\left(e^{-x}\right)$$

$$= \frac{\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}} \cdot e^{-x} \cdot \frac{d}{dx}(-x)$$

$$= \frac{e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}} \times (-1)$$

$$= \frac{-e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}$$

Question 5:

Differentiate the following w.r.t. x:

$\log\left(\cos e^{x}\right)$

Answer

Let $y = \log\left(\cos e^x\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\cos e^x\right)\right]$$

$$= \frac{1}{\cos e^x} \cdot \frac{d}{dx}\left(\cos e^x\right)$$

$$= \frac{1}{\cos e^x} \cdot \left(-\sin e^x\right) \cdot \frac{d}{dx}\left(e^x\right)$$

$$= \frac{-\sin e^x}{\cos e^x} \cdot e^x$$

$$= -e^x \tan e^x, e^x \neq (2n+1)\frac{\pi}{2}, n \in \mathbf{N}$$

Question 6:

Differentiate the following w.r.t. x:

$e^x + e^{x^2} + ... + e^{x^5}$

Answer

$$\frac{d}{dx}\left(e^x + e^{x^2} + ... + e^{x^5}\right)$$

$$= \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(e^{x^2}\right) + \frac{d}{dx}\left(e^{x^3}\right) + \frac{d}{dx}\left(e^{x^4}\right) + \frac{d}{dx}\left(e^{x^5}\right)$$

$$= e^x + \left[e^{x^2} \times \frac{d}{dx}\left(x^2\right)\right] + \left[e^{x^3} \cdot \frac{d}{dx}\left(x^3\right)\right] + \left[e^{x^4} \cdot \frac{d}{dx}\left(x^4\right)\right] + \left[e^{x^5} \cdot \frac{d}{dx}\left(x^5\right)\right]$$

$$= e^x + \left(e^{x^2} \times 2x\right) + \left(e^{x^3} \times 3x^2\right) + \left(e^{x^4} \times 4x^3\right) + \left(e^{x^5} \times 5x^4\right)$$

$$= e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5}$$

Question 7:

Differentiate the following w.r.t. x:

$\sqrt{e^{\sqrt{x}}}, x > 0$

Answer

Let $y = \sqrt{e^{\sqrt{x}}}$

Then, $y^2 = e^{\sqrt{x}}$

By differentiating this relationship with respect to x, we obtain

$$y^2 = e^{\sqrt{x}}$$

$$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{d}{dx}\left(\sqrt{x}\right) \qquad \text{[By applying the chain rule]}$$

$$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{1}{2}\cdot\frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4y\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}, \ x > 0$$

Question 8:

Differentiate the following w.r.t. x:

$\log(\log x), x > 1$

Answer

Let $y = \log(\log x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(\log x)\right]$$

$$= \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$

$$= \frac{1}{\log x} \cdot \frac{1}{x}$$

$= \frac{1}{x \log x}$, x > 1

Question 9:

Differentiate the following w.r.t. x:

$\frac{\cos x}{\log x}, x > 0$

Answer

Let $y=\frac{\cos x}{\log x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx}=\frac{\frac{d}{dx}(\cos x)\times\log x-\cos x\times\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$=\frac{-\sin x\log x-\cos x\times\frac{1}{x}}{(\log x)^2}$$

$$=\frac{-[x\log x.\sin x+\cos x]}{x(\log x)^2},x>0$$

Question 10:

Differentiate the following w.r.t. x:

$\cos\left(\log x + e^x\right), x > 0$

Answer

Let $y = \cos\left(\log x + e^x\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = -\sin\left(\log x + e^x\right) \cdot \frac{d}{dx}\left(\log x + e^x\right)$$

$$= -\sin\left(\log x + e^x\right) \cdot \left[\frac{d}{dx}(\log x) + \frac{d}{dx}\left(e^x\right)\right]$$

$$= -\sin\left(\log x + e^x\right) \cdot \left(\frac{1}{x} + e^x\right)$$

$$= -\left(\frac{1}{x} + e^x\right)\sin\left(\log x + e^x\right), x > 0$$

Exercise 5.5

Question 1:

Differentiate the function with respect to x.

$\cos x.\cos 2x.\cos 3x$

Answer

Let $y = \cos x.\cos 2x.\cos 3x$

Taking logarithm on both the sides, we obtain

$$\log y = \log\left(\cos x.\cos 2x.\cos 3x\right)$$
$$\Rightarrow \log y = \log\left(\cos x\right) + \log\left(\cos 2x\right) + \log\left(\cos 3x\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos x}\cdot\frac{d}{dx}\left(\cos x\right) + \frac{1}{\cos 2x}\cdot\frac{d}{dx}\left(\cos 2x\right) + \frac{1}{\cos 3x}\cdot\frac{d}{dx}\left(\cos 3x\right)$$
$$\Rightarrow \frac{dy}{dx} = y\left[-\frac{\sin x}{\cos x} - \frac{\sin 2x}{\cos 2x}\cdot\frac{d}{dx}\left(2x\right) - \frac{\sin 3x}{\cos 3x}\cdot\frac{d}{dx}\left(3x\right)\right]$$
$$\therefore \frac{dy}{dx} = -\cos x.\cos 2x.\cos 3x\left[\tan x + 2\tan 2x + 3\tan 3x\right]$$

Question 2:

Differentiate the function with respect to x.

$$\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Answer

$$\text{Let } y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Taking logarithm on both the sides, we obtain

$$\log y = \log\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

$$\Rightarrow \log y = \frac{1}{2}\log\left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]$$

$$\Rightarrow \log y = \frac{1}{2}\left[\log\{(x-1)(x-2)\} - \log\{(x-3)(x-4)(x-5)\}\right]$$

$$\Rightarrow \log y = \frac{1}{2}\left[\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4)-\log(x-5)\right]$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\begin{array}{l}\frac{1}{x-1}\cdot\frac{d}{dx}(x-1)+\frac{1}{x-2}\cdot\frac{d}{dx}(x-2)-\frac{1}{x-3}\cdot\frac{d}{dx}(x-3)\\ -\frac{1}{x-4}\cdot\frac{d}{dx}(x-4)-\frac{1}{x-5}\cdot\frac{d}{dx}(x-5)\end{array}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2}\left(\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right)$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$$

Question 3:

Differentiate the function with respect to x.

$$(\log x)^{\cos x}$$

Answer

Let $y = (\log x)^{\cos x}$

Taking logarithm on both the sides, we obtain

$$\log y = \cos x \cdot \log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}(\cos x)\times\log(\log x)+\cos x\times\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow\frac{1}{y}.\frac{dy}{dx}=-\sin x\log(\log x)+\cos x\times\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{dy}{dx}=y\left[-\sin x\log(\log x)+\frac{\cos x}{\log x}\times\frac{1}{x}\right]$$

$$\therefore\frac{dy}{dx}=(\log x)^{\cos x}\left[\frac{\cos x}{x\log x}-\sin x\log(\log x)\right]$$

Question 4:

Differentiate the function with respect to x.

$x^x-2^{\sin x}$

Answer

Let $y=x^x-2^{\sin x}$

Also, let $x^x=u$ and $2^{\sin x}=v$

$\therefore y=u-v$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx}$$

$u = x^x$

Taking logarithm on both the sides, we obtain

$$\log u = x \log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \left[\frac{d}{dx}(x) \times \log x + x \times \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1 \times \log x + x \times \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^x(\log x + 1)$$

$$\Rightarrow \frac{du}{dx} = x^x(1 + \log x)$$

$v = 2^{\sin x}$

Taking logarithm on both the sides with respect to x, we obtain

$$\log v = \sin x \cdot \log 2$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log 2 \cdot \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{dv}{dx} = v \log 2 \cos x$$

$$\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2$$

$$\therefore \frac{dy}{dx} = x^x(1 + \log x) - 2^{\sin x} \cos x \log 2$$

Question 5:

Differentiate the function with respect to x.

$(x+3)^2.(x+4)^3.(x+5)^4$

Answer

Let $y=(x+3)^2.(x+4)^3.(x+5)^4$

Taking logarithm on both the sides, we obtain

$$\log y=\log(x+3)^2+\log(x+4)^3+\log(x+5)^4$$

$$\Rightarrow \log y=2\log(x+3)+3\log(x+4)+4\log(x+5)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}.\frac{dy}{dx}=2\cdot\frac{1}{x+3}\cdot\frac{d}{dx}(x+3)+3\cdot\frac{1}{x+4}\cdot\frac{d}{dx}(x+4)+4\cdot\frac{1}{x+5}\cdot\frac{d}{dx}(x+5)$$

$$\Rightarrow \frac{dy}{dx}=y\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx}=(x+3)^2(x+4)^3(x+5)^4\cdot\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx}=(x+3)^2(x+4)^3(x+5)^4\cdot\left[\frac{2(x+4)(x+5)+3(x+3)(x+5)+4(x+3)(x+4)}{(x+3)(x+4)(x+5)}\right]$$

$$\Rightarrow \frac{dy}{dx}=(x+3)(x+4)^2(x+5)^3\cdot\left[2(x^2+9x+20)+3(x^2+8x+15)+4(x^2+7x+12)\right]$$

$$\therefore \frac{dy}{dx}=(x+3)(x+4)^2(x+5)^3(9x^2+70x+133)$$

Question 6:

Differentiate the function with respect to x.

$$\left(x+\frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$$

Answer

Let $y = \left(x+\frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Also, let $u = \left(x+\frac{1}{x}\right)^x$ and $v = x^{\left(1+\frac{1}{x}\right)}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots(1)$$

Then, $u = \left(x+\frac{1}{x}\right)^x$

$$\Rightarrow \log u = \log\left(x+\frac{1}{x}\right)^x$$

$$\Rightarrow \log u = x\log\left(x+\frac{1}{x}\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}(x)\times\log\left(x+\frac{1}{x}\right)+x\times\frac{d}{dx}\left[\log\left(x+\frac{1}{x}\right)\right]$$

$$\Rightarrow\frac{1}{u}\frac{du}{dx}=1\times\log\left(x+\frac{1}{x}\right)+x\times\frac{1}{\left(x+\frac{1}{x}\right)}\cdot\frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$\Rightarrow\frac{du}{dx}=u\left[\log\left(x+\frac{1}{x}\right)+\frac{x}{\left(x+\frac{1}{x}\right)}\times\left(1-\frac{1}{x^2}\right)\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{\left(x-\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)}\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{x^2-1}{x^2+1}\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right] \qquad ...(2)$$

$$v=x^{\left(1+\frac{1}{x}\right)}$$

$$\Rightarrow\log v=\log\left[x^{\left(1+\frac{1}{x}\right)}\right]$$

$$\Rightarrow\log v=\left(1+\frac{1}{x}\right)\log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\left[\frac{d}{dx}\left(1+\frac{1}{x}\right)\right]\times\log x+\left(1+\frac{1}{x}\right)\cdot\frac{d}{dx}\log x$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\left(-\frac{1}{x^2}\right)\log x+\left(1+\frac{1}{x}\right)\cdot\frac{1}{x}$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=-\frac{\log x}{x^2}+\frac{1}{x}+\frac{1}{x^2}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{-\log x+x+1}{x^2}\right]$$

$$\Rightarrow\frac{dv}{dx}=x^{\left(1+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right)\qquad ...(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]+x^{\left(1+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right)$$

Question 7:

Differentiate the function with respect to x.

$(\log x)^x + x^{\log x}$

Answer

Let $y = (\log x)^x + x^{\log x}$

Also, let $u = (\log x)^x$ and $v = x^{\log x}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(1)$$

u = (log x)^x

$\Rightarrow \log u = \log\left[(\log x)^x\right]$

$\Rightarrow \log u = x \log(\log x)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\times\log(\log x)+x\cdot\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=u\left[1\times\log(\log x)+x\cdot\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^x\left[\log(\log x)+\frac{x}{\log x}\cdot\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^x\left[\log(\log x)+\frac{1}{\log x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^x\left[\frac{\log(\log x)\cdot\log x+1}{\log x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right]\qquad ...(2)$$

$$v=x^{\log x}$$

$$\Rightarrow\log v=\log\left(x^{\log x}\right)$$

$$\Rightarrow\log v=\log x\log x=(\log x)^2$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\frac{d}{dx}\left[(\log x)^2\right]$$

$$\Rightarrow\frac{1}{v}.\frac{dv}{dx}=2(\log x)\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{dv}{dx}=2v(\log x)\cdot\frac{1}{x}$$

$$\Rightarrow\frac{dv}{dx}=2x^{\log x}\frac{\log x}{x}$$

$$\Rightarrow\frac{dv}{dx}=2x^{\log x-1}\cdot\log x\qquad ...(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (\log x)^{x-1}\left[1 + \log x.\log(\log x)\right] + 2x^{\log x-1} \cdot \log x$$

Question 8:

Differentiate the function with respect to x.

Answer

Let $y=(\sin x)^x+\sin^{-1}\sqrt{x}$

Also, let $u=(\sin x)^x$ and $v=\sin^{-1}\sqrt{x}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$u=(\sin x)^x$

$\Rightarrow \log u=\log(\sin x)^x$

$\Rightarrow \log u=x\log(\sin x)$

Differentiating both sides with respect to x, we obtain

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\times\log(\sin x)+x\times\frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx}=u\left[1\cdot\log(\sin x)+x\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx}=(\sin x)^x\left[\log(\sin x)+\frac{x}{\sin x}\cdot\cos x\right]$$

$$\Rightarrow \frac{du}{dx}=(\sin x)^x(x\cot x+\log\sin x) \qquad ...(2)$$

$v=\sin^{-1}\sqrt{x}$

Differentiating both sides with respect to x, we obtain

$$\frac{dv}{dx}=\frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$\Rightarrow \frac{dv}{dx}=\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx}=\frac{1}{2\sqrt{x-x^2}} \qquad ...(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx}=(\sin x)^x(x\cot x+\log\sin x)+\frac{1}{2\sqrt{x-x^2}}$$

Question 9:

Differentiate the function with respect to x.

$$x^{\sin x}+(\sin x)^{\cos x}$$

Answer

$$\text{Let } y=x^{\sin x}+(\sin x)^{\cos x}$$

$$\text{Also, let } u=x^{\sin x} \text{ and } v=(\sin x)^{\cos x}$$

$$\therefore y=u+v$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$$u=x^{\sin x}$$

$$\Rightarrow \log u=\log\left(x^{\sin x}\right)$$

$$\Rightarrow \log u=\sin x\log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(\sin x)\cdot\log x+\sin x\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx}=u\left[\cos x\log x+\sin x\cdot\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx}=x^{\sin x}\left[\cos x\log x+\frac{\sin x}{x}\right] \qquad ...(2)$$

$$v=(\sin x)^{\cos x}$$

$$\Rightarrow \log v=\log(\sin x)^{\cos x}$$

$$\Rightarrow \log v=\cos x\log(\sin x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}(\cos x)\times\log(\sin x) + \cos x \times \frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow \frac{dv}{dx} = v\left[-\sin x.\log(\sin x) + \cos x \cdot \frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[-\sin x\log\sin x + \frac{\cos x}{\sin x}\cos x\right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[-\sin x\log\sin x + \cot x\cos x\right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[\cot x\cos x - \sin x\log\sin x\right] \quad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = x^{\sin x}\left(\cos x\log x + \frac{\sin x}{x}\right) + (\sin x)^{\cos x}\left[\cos x\cot x - \sin x\log\sin x\right]$$

Question 10:

Differentiate the function with respect to x.

$$x^{x\cos x}+\frac{x^2+1}{x^2-1}$$

Answer

Let $y=x^{x\cos x}+\dfrac{x^2+1}{x^2-1}$

Also, let $u=x^{x\cos x}$ and $v=\dfrac{x^2+1}{x^2-1}$

$\therefore\ y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$u=x^{x\cos x}$

$\Rightarrow \log u=\log\left(x^{x\cos x}\right)$

$\Rightarrow \log u=x\cos x\log x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\cdot\cos x\cdot\log x+x\cdot\frac{d}{dx}(\cos x)\cdot\log x+x\cos x\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{du}{dx}=u\left[1\cdot\cos x\cdot\log x+x\cdot(-\sin x)\log x+x\cos x\cdot\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=x^{x\cos x}\left(\cos x\log x-x\sin x\log x+\cos x\right)$$

$$\Rightarrow\frac{du}{dx}=x^{x\cos x}\left[\cos x(1+\log x)-x\sin x\log x\right]\qquad ...(2)$$

$$v=\frac{x^2+1}{x^2-1}$$

$$\Rightarrow\log v=\log\left(x^2+1\right)-\log\left(x^2-1\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{2x}{x^2+1}-\frac{2x}{x^2-1}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{2x\left(x^2-1\right)-2x\left(x^2+1\right)}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow\frac{dv}{dx}=\frac{x^2+1}{x^2-1}\times\left[\frac{-4x}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow\frac{dv}{dx}=\frac{-4x}{\left(x^2-1\right)^2}\qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx}=x^{x\cos x}\left[\cos x(1+\log x)-x\sin x\log x\right]-\frac{4x}{\left(x^2-1\right)^2}$$

Question 11:

Differentiate the function with respect to x.

$$(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$$

Answer

Let $y=(x\cos x)^x+(x\sin x)^{\frac{1}{x}}$

Also, let $u=(x\cos x)^x$ and $v=(x\sin x)^{\frac{1}{x}}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$u=(x\cos x)^x$

$\Rightarrow \log u=\log(x\cos x)^x$

$\Rightarrow \log u=x\log(x\cos x)$

$\Rightarrow \log u=x[\log x+\log\cos x]$

$\Rightarrow \log u=x\log x+x\log\cos x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x\log x)+\frac{d}{dx}(x\log\cos x)$$

$$\Rightarrow \frac{du}{dx}=u\left[\left\{\log x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log x)\right\}+\left\{\log\cos x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x\left[\left(\log x\cdot 1+x\cdot\frac{1}{x}\right)+\left\{\log\cos x\cdot 1+x\cdot\frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x\left[(\log x+1)+\left\{\log\cos x+\frac{x}{\cos x}\cdot(-\sin x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x\left[(1+\log x)+(\log\cos x-x\tan x)\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x\left[1-x\tan x+(\log x+\log\cos x)\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x\left[1-x\tan x+\log(x\cos x)\right] \qquad ...(2)$$

$$v = (x\sin x)^{\frac{1}{x}}$$
$$\Rightarrow \log v = \log (x\sin x)^{\frac{1}{x}}$$
$$\Rightarrow \log v = \frac{1}{x}\log (x\sin x)$$
$$\Rightarrow \log v = \frac{1}{x}(\log x + \log \sin x)$$
$$\Rightarrow \log v = \frac{1}{x}\log x + \frac{1}{x}\log \sin x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{x}\log x\right) + \frac{d}{dx}\left[\frac{1}{x}\log(\sin x)\right]$$
$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x \cdot \frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}\cdot\frac{d}{dx}(\log x)\right] + \left[\log(\sin x)\cdot\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}\cdot\frac{d}{dx}\{\log(\sin x)\}\right]$$
$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x \cdot\left(-\frac{1}{x^2}\right) + \frac{1}{x}\cdot\frac{1}{x}\right] + \left[\log(\sin x)\cdot\left(-\frac{1}{x^2}\right) + \frac{1}{x}\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$
$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2}(1 - \log x) + \left[-\frac{\log(\sin x)}{x^2} + \frac{1}{x\sin x}\cdot\cos x\right]$$
$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x}{x^2} + \frac{-\log(\sin x) + x\cot x}{x^2}\right]$$
$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x - \log(\sin x) + x\cot x}{x^2}\right]$$
$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log(x\sin x) + x\cot x}{x^2}\right] \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (x\cos x)^x\left[1 - x\tan x + \log(x\cos x)\right] + (x\sin x)^{\frac{1}{x}}\left[\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right]$$

Question 12:

Find $\frac{dy}{dx}$ of function.

$x^y + y^x = 1$

Answer

The given function is $x^y + y^x = 1$

Let $x^y = u$ and $y^x = v$

Then, the function becomes u + v = 1

$$\therefore \frac{du}{dx}+\frac{dv}{dx}=0 \qquad ...(1)$$

$$u=x^y$$

$$\Rightarrow \log u=\log\left(x^y\right)$$

$$\Rightarrow \log u=y\log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\log x\frac{dy}{dx}+y\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx}=u\left[\log x\frac{dy}{dx}+y\cdot\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx}=x^y\left(\log x\frac{dy}{dx}+\frac{y}{x}\right) \qquad ...(2)$$

$$v=y^x$$

$$\Rightarrow \log v=\log\left(y^x\right)$$

$$\Rightarrow \log v=x\log y$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\log y\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log y)$$

$$\Rightarrow \frac{dv}{dx}=v\left(\log y\cdot 1+x\cdot\frac{1}{y}\cdot\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dv}{dx}=y^x\left(\log y+\frac{x}{y}\frac{dy}{dx}\right) \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$x^y\left(\log x\frac{dy}{dx}+\frac{y}{x}\right)+y^x\left(\log y+\frac{x}{y}\frac{dy}{dx}\right)=0$$

$$\Rightarrow\left(x^y\log x+xy^{x-1}\right)\frac{dy}{dx}=-\left(yx^{y-1}+y^x\log y\right)$$

$$\therefore\frac{dy}{dx}=-\frac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$$

Question 13:

Find $\frac{dy}{dx}$

$y^x=x^y$ of function.

Answer The given function is

$$y^x=x^y$$

Taking logarithm on both the sides, we obtain

$x\log y = y\log x$

Differentiating both sides with respect to x, we obtain

$$\log y\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log y)=\log x\cdot\frac{d}{dx}(y)+y\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \log y\cdot 1+x\cdot\frac{1}{y}\cdot\frac{dy}{dx}=\log x\cdot\frac{dy}{dx}+y\cdot\frac{1}{x}$$

$$\Rightarrow \log y+\frac{x}{y}\frac{dy}{dx}=\log x\frac{dy}{dx}+\frac{y}{x}$$

$$\Rightarrow \left(\frac{x}{y}-\log x\right)\frac{dy}{dx}=\frac{y}{x}-\log y$$

$$\Rightarrow \left(\frac{x-y\log x}{y}\right)\frac{dy}{dx}=\frac{y-x\log y}{x}$$

$$\therefore \frac{dy}{dx}=\frac{y}{x}\left(\frac{y-x\log y}{x-y\log x}\right)$$

Question 14:

Find $\frac{dy}{dx}$ of function.

$(\cos x)^y=(\cos y)^x$

Answer

The given function is $(\cos x)^y = (\cos y)^x$

Taking logarithm on both the sides, we obtain

$$y \log \cos x = x \log \cos y$$

Differentiating both sides, we obtain

$$\log \cos x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log \cos x) = \log \cos y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log \cos y)$$

$$\Rightarrow \log \cos x \frac{dy}{dx} + y \cdot \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) = \log \cos y \cdot 1 + x \cdot \frac{1}{\cos y} \cdot \frac{d}{dx}(\cos y)$$

$$\Rightarrow \log \cos x \frac{dy}{dx} + \frac{y}{\cos x} \cdot (-\sin x) = \log \cos y + \frac{x}{\cos y}(-\sin y) \cdot \frac{dy}{dx}$$

$$\Rightarrow \log \cos x \frac{dy}{dx} - y \tan x = \log \cos y - x \tan y \frac{dy}{dx}$$

$$\Rightarrow (\log \cos x + x \tan y)\frac{dy}{dx} = y \tan x + \log \cos y$$

$$\therefore \frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}$$

Question 15:

Find $\frac{dy}{dx}$ of function.

$$xy = e^{(x-y)}$$

Answer

The given function is $xy = e^{(x-y)}$

Taking logarithm on both the sides, we obtain

$$\log(xy) = \log\left(e^{x-y}\right)$$

$$\Rightarrow \log x + \log y = (x-y)\log e$$

$$\Rightarrow \log x + \log y = (x-y)\times 1$$

$$\Rightarrow \log x + \log y = x - y$$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}(\log x) + \frac{d}{dx}(\log y) = \frac{d}{dx}(x) - \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \left(1 + \frac{1}{y}\right)\frac{dy}{dx} = 1 - \frac{1}{x}$$

$$\Rightarrow \left(\frac{y+1}{y}\right)\frac{dy}{dx} = \frac{x-1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$$

Question 16:

Find the derivative of the function given by $f(x) = (1+x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^8\right)$ and hence find $f'(1)$.

Answer

The given relationship is $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$

Taking logarithm on both the sides, we obtain

$$\log f(x)=\log(1+x)+\log(1+x^2)+\log(1+x^4)+\log(1+x^8)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{f(x)}\cdot\frac{d}{dx}[f(x)]=\frac{d}{dx}\log(1+x)+\frac{d}{dx}\log(1+x^2)+\frac{d}{dx}\log(1+x^4)+\frac{d}{dx}\log(1+x^8)$$

$$\Rightarrow \frac{1}{f(x)}\cdot f'(x)=\frac{1}{1+x}\cdot\frac{d}{dx}(1+x)+\frac{1}{1+x^2}\cdot\frac{d}{dx}(1+x^2)+\frac{1}{1+x^4}\cdot\frac{d}{dx}(1+x^4)+\frac{1}{1+x^8}\cdot\frac{d}{dx}(1+x^8)$$

$$\Rightarrow f'(x)=f(x)\left[\frac{1}{1+x}+\frac{1}{1+x^2}.2x+\frac{1}{1+x^4}\cdot 4x^3+\frac{1}{1+x^8}\cdot 8x^7\right]$$

$$\therefore f'(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right]$$

$$\text{Hence, } f'(1)=(1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1}+\frac{2\times1}{1+1^2}+\frac{4\times1^3}{1+1^4}+\frac{8\times1^7}{1+1^8}\right]$$

$$=2\times2\times2\times2\left[\frac{1}{2}+\frac{2}{2}+\frac{4}{2}+\frac{8}{2}\right]$$

$$=16\times\left(\frac{1+2+4+8}{2}\right)$$

$$=16\times\frac{15}{2}=120$$

Question 17:

Differentiate $(x^5-5x+8)(x^3+7x+9)$ in three ways mentioned below

(i) By using product rule.

(ii) By expanding the product to obtain a single polynomial.

(iii By logarithmic differentiation.

Do they all give the same answer?

Answer

(i) Let $y = \left(x^5 - 5x + 8\right)\left(x^3 + 7x + 9\right)$

Let $x^2 - 5x + 8 = u$ and $x^3 + 7x + 9 = v$

$\therefore y = uv$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx} \qquad \text{(By using product rule)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(x^2 - 5x + 8\right) \cdot \left(x^3 + 7x + 9\right) + \left(x^2 - 5x + 8\right) \cdot \frac{d}{dx}\left(x^3 + 7x + 9\right)$$

$$\Rightarrow \frac{dy}{dx} = (2x - 5)\left(x^3 + 7x + 9\right) + \left(x^2 - 5x + 8\right)\left(3x^2 + 7\right)$$

$$\Rightarrow \frac{dy}{dx} = 2x\left(x^3 + 7x + 9\right) - 5\left(x^3 + 7x + 9\right) + x^2\left(3x^2 + 7\right) - 5x\left(3x^2 + 7\right) + 8\left(3x^2 + 7\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(2x^4 + 14x^2 + 18x\right) - 5x^3 - 35x - 45 + \left(3x^4 + 7x^2\right) - 15x^3 - 35x + 24x^2 + 56$$

$$\therefore \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

(ii) $y = \left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)$

$$= x^2\left(x^3 + 7x + 9\right) - 5x\left(x^3 + 7x + 9\right) + 8\left(x^3 + 7x + 9\right)$$

$$= x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$$

$$= x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72\right)$$

$$= \frac{d}{dx}\left(x^5\right) - 5\frac{d}{dx}\left(x^4\right) + 15\frac{d}{dx}\left(x^3\right) - 26\frac{d}{dx}\left(x^2\right) + 11\frac{d}{dx}(x) + \frac{d}{dx}(72)$$

$$= 5x^4 - 5 \times 4x^3 + 15 \times 3x^2 - 26 \times 2x + 11 \times 1 + 0$$

$$= 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

(iii) $y = \left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)$

Taking logarithm on both the sides, we obtain

$$\log y = \log\left(x^2 - 5x + 8\right) + \log\left(x^3 + 7x + 9\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\log\left(x^2-5x+8\right)+\frac{d}{dx}\log\left(x^3+7x+9\right)$$

$$\Rightarrow\frac{1}{y}\frac{dy}{dx}=\frac{1}{x^2-5x+8}\cdot\frac{d}{dx}\left(x^2-5x+8\right)+\frac{1}{x^3+7x+9}\cdot\frac{d}{dx}\left(x^3+7x+9\right)$$

$$\Rightarrow\frac{dy}{dx}=y\left[\frac{1}{x^2-5x+8}\times(2x-5)+\frac{1}{x^3+7x+9}\times\left(3x^2+7\right)\right]$$

$$\Rightarrow\frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right]$$

$$\Rightarrow\frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{(2x-5)\left(x^3+7x+9\right)+\left(3x^2+7\right)\left(x^2-5x+8\right)}{\left(x^2-5x+8\right)\left(x^3+7x+9\right)}\right]$$

$$\Rightarrow\frac{dy}{dx}=2x\left(x^3+7x+9\right)-5\left(x^3+7x+9\right)+3x^2\left(x^2-5x+8\right)+7\left(x^2-5x+8\right)$$

$$\Rightarrow\frac{dy}{dx}=\left(2x^4+14x^2+18x\right)-5x^3-35x-45+\left(3x^4-15x^3+24x^2\right)+\left(7x^2-35x+56\right)$$

$$\Rightarrow\frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

From the above three observations, it can be concluded that all the results of $\frac{dy}{dx}$ are same.

Question 18:

If u, v and w are functions of x, then show that

$$\frac{d}{dx}(u.v.w)=\frac{du}{dx}v.w+u.\frac{dv}{dx}.w+u.v.\frac{dw}{dx}$$

in two ways-first by repeated application of product rule, second by logarithmic differentiation.

Answer

Let $y = u.v.w = u.(v.w)$

By applying product rule, we obtain

$$\frac{dy}{dx} = \frac{du}{dx}.(v.w) + u.\frac{d}{dx}(v \cdot w)$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} v \cdot w + u\left[\frac{dv}{dx} \cdot w + v \cdot \frac{dw}{dx}\right] \qquad \text{(Again applying product rule)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$$

By taking logarithm on both sides of the equation $y = u.v.w$, we obtain

$$\log y = \log u + \log v + \log w$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(\log u) + \frac{d}{dx}(\log v) + \frac{d}{dx}(\log w)$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = u.v.w.\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$$

Exercise 5.6

Question 1:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x = 2at^2,\ y = at^4$.

Answer

The given equations are $x = 2at^2$ and $y = at^4$

Then, $\frac{dx}{dt} = \frac{d}{dt}\left(2at^2\right) = 2a \cdot \frac{d}{dt}\left(t^2\right) = 2a \cdot 2t = 4at$

$\frac{dy}{dt} = \frac{d}{dt}\left(at^4\right) = a \cdot \frac{d}{dt}\left(t^4\right) = a \cdot 4 \cdot t^3 = 4at^3$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4at^3}{4at} = t^2$

Question 2:

If x and y are connected parametrically by the equation, without eliminating the

parameter, find $\frac{dy}{dx}$.

x = a cos θ, y = b cos θ

Answer

The given equations are x = a cos θ and y = b cos θ

$$\text{Then, } \frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos\theta) = a(-\sin\theta) = -a\sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b\cos\theta) = b(-\sin\theta) = -b\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$$

Question 3:

If x and y are connected parametrically by the equation, without eliminating the

parameter, find $\frac{dy}{dx}$. x

= sin t, y = cos 2t

Answer

The given equations are x = sin t and y = cos 2t

Then, $\frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos 2t) = -\sin 2t \cdot \frac{d}{dt}(2t) = -2\sin 2t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{\cos t} = \frac{-2 \cdot 2\sin t\cos t}{\cos t} = -4\sin t$$

Question 4:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = 4t,\ y = \frac{4}{t}$

Answer

The given equations are $x = 4t$ and $y = \frac{4}{t}$

$$\frac{dx}{dt} = \frac{d}{dt}(4t) = 4$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right) = 4 \cdot \frac{d}{dt}\left(\frac{1}{t}\right) = 4 \cdot \left(\frac{-1}{t^2}\right) = \frac{-4}{t^2}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-4}{t^2}\right)}{4} = \frac{-1}{t^2}$$

Question 5:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x=\cos\theta-\cos 2\theta,\ y=\sin\theta-\sin 2\theta$

Answer

The given equations are $x=\cos\theta-\cos 2\theta$ and $y=\sin\theta-\sin 2\theta$

$$\text{Then, } \frac{dx}{d\theta}=\frac{d}{d\theta}(\cos\theta-\cos 2\theta)=\frac{d}{d\theta}(\cos\theta)-\frac{d}{d\theta}(\cos 2\theta)$$

$$=-\sin\theta-(-2\sin 2\theta)=2\sin 2\theta-\sin\theta$$

$$\frac{dy}{d\theta}=\frac{d}{d\theta}(\sin\theta-\sin 2\theta)=\frac{d}{d\theta}(\sin\theta)-\frac{d}{d\theta}(\sin 2\theta)$$

$$=\cos\theta-2\cos 2\theta$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{\cos\theta-2\cos 2\theta}{2\sin 2\theta-\sin\theta}$$

Question 6:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x=a(\theta-\sin\theta),\ y=a(1+\cos\theta)$

Answer

The given equations are $x = a(\theta - \sin\theta)$ and $y = a(1+\cos\theta)$

$$\text{Then, } \frac{dx}{d\theta} = a\left[\frac{d}{d\theta}(\theta) - \frac{d}{d\theta}(\sin\theta)\right] = a(1-\cos\theta)$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}(\cos\theta)\right] = a\left[0 + (-\sin\theta)\right] = -a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

Question 7:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$$x = \frac{\sin^3 t}{\sqrt{\cos 2t}},\ y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$$

Answer

The given equations are $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

Cla Then, $\frac{dx}{dt}=\frac{d}{dt}\left[\frac{\sin^3 t}{\sqrt{\cos 2t}}\right]$

$$=\frac{\sqrt{\cos 2t}\cdot\frac{d}{dt}\left(\sin^3 t\right)-\sin^3 t\cdot\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}\cdot 3\sin^2 t\cdot\frac{d}{dt}(\sin t)-\sin^3 t\times\frac{1}{2\sqrt{\cos 2t}}.\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$=\frac{3\sqrt{\cos 2t}\cdot\sin^2 t\cos t-\frac{\sin^3 t}{2\sqrt{\cos 2t}}.(-2\sin 2t)}{\cos 2t}$$

$$=\frac{3\cos 2t\sin^2 t\cos t+\sin^3 t\sin 2t}{\cos 2t\sqrt{\cos 2t}}$$

$$\frac{dy}{dt}=\frac{d}{dt}\left[\frac{\cos^3 t}{\sqrt{\cos 2t}}\right]$$

$$=\frac{\sqrt{\cos 2t}.\frac{d}{dt}\left(\cos^3 t\right)-\cos^3 t\cdot\frac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}.3\cos^2 t\cdot\frac{d}{dt}(\cos t)-\cos^3 t.\frac{1}{2\sqrt{\cos 2t}}\cdot\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$=\frac{3\sqrt{\cos 2t}.\cos^2 t(-\sin t)-\cos^3 t\cdot\frac{1}{2\sqrt{\cos 2t}}.(-2\sin 2t)}{\cos 2t}$$

$$=\frac{-3\cos 2t.\cos^2 t.\sin t+\cos^3 t\sin 2t}{\cos 2t\cdot\sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t \sin 2t}{3\cos 2t \sin^2 t \cos t + \sin^3 t \sin 2t}$$

$$= \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t(2\sin t \cos t)}{3\cos 2t \sin^2 t \cos t + \sin^3 t(2\sin t \cos t)}$$

$$= \frac{\sin t \cos t\left[-3\cos 2t.\cos t + 2\cos^3 t\right]}{\sin t \cos t\left[3\cos 2t \sin t + 2\sin^3 t\right]}$$

$$= \frac{\left[-3(2\cos^2 t - 1)\cos t + 2\cos^3 t\right]}{\left[3(1 - 2\sin^2 t)\sin t + 2\sin^3 t\right]} \qquad \begin{bmatrix}\cos 2t = (2\cos^2 t - 1), \\ \cos 2t = (1 - 2\sin^2 t)\end{bmatrix}$$

$$= \frac{-4\cos^3 t + 3\cos t}{3\sin t - 4\sin^3 t}$$

$$= \frac{-\cos 3t}{\sin 3t} \qquad \begin{bmatrix}\cos 3t = 4\cos^3 t - 3\cos t, \\ \sin 3t = 3\sin t - 4\sin^3 t\end{bmatrix}$$

$$= -\cot 3t$$

Question 8:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$$x = a\left(\cos t + \log \tan \frac{t}{2}\right),\ y = a\sin t$$

Answer

The given equations are $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$ and $y = a\sin t$

$$\text{Then, } \frac{dx}{dt} = a \cdot \left[\frac{d}{dt}(\cos t) + \frac{d}{dt}\left(\log \tan \frac{t}{2} \right) \right]$$

$$= a \left[-\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \frac{d}{dt}\left(\tan \frac{t}{2} \right) \right]$$

$$= a \left[-\sin t + \cot \frac{t}{2} \cdot \sec^2 \frac{t}{2} \cdot \frac{d}{dt}\left(\frac{t}{2} \right) \right]$$

$$= a \left[-\sin t + \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} \times \frac{1}{\cos^2 \frac{t}{2}} \times \frac{1}{2} \right]$$

$$= a \left[-\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}} \right]$$

$$= a \left(-\sin t + \frac{1}{\sin t} \right)$$

$$= a \left(\frac{-\sin^2 t + 1}{\sin t} \right)$$

$$= a \frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a \frac{d}{dt}(\sin t) = a \cos t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt} \right)}{\left(\frac{dx}{dt} \right)} = \frac{a \cos t}{\left(a \frac{\cos^2 t}{\sin t} \right)} = \frac{\sin t}{\cos t} = \tan t$$

Question 9:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a\sec\theta,\ y = b\tan\theta$

Answer

The given equations are $x = a\sec\theta$ and $y = b\tan\theta$

Then, $\frac{dx}{d\theta} = a\cdot\frac{d}{d\theta}(\sec\theta) = a\sec\theta\tan\theta$

$\frac{dy}{d\theta} = b\cdot\frac{d}{d\theta}(\tan\theta) = b\sec^2\theta$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\sec\theta\cot\theta = \frac{b\cos\theta}{a\cos\theta\sin\theta} = \frac{b}{a}\times\frac{1}{\sin\theta} = \frac{b}{a}\operatorname{cosec}\theta$$

Question 10:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$

Answer

The given equations are $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

Then, $\frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\cos\theta + \frac{d}{d\theta}(\theta\sin\theta)\right] = a\left[-\sin\theta + \theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta)\right]$

$= a[-\sin\theta + \theta\cos\theta + \sin\theta] = a\theta\cos\theta$

$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right] = a\left[\cos\theta - \left\{\theta\frac{d}{d\theta}(\cos\theta) + \cos\theta\cdot\frac{d}{d\theta}(\theta)\right\}\right]$

$= a[\cos\theta + \theta\sin\theta - \cos\theta]$

$= a\theta\sin\theta$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$

Question 11:

If $x = \sqrt{a^{\sin^{-1}t}},\ y = \sqrt{a^{\cos^{-1}t}},$ show that $\frac{dy}{dx} = -\frac{y}{x}$

Answer

The given equations are $x=\sqrt{a^{\sin^{-1}t}}$ and $y=\sqrt{a^{\cos^{-1}t}}$

$x=\sqrt{a^{\sin^{-1}t}}$ and $y=\sqrt{a^{\cos^{-1}t}}$

$\Rightarrow x=\left(a^{\sin^{-1}t}\right)^{\frac{1}{2}}$ and $y=\left(a^{\cos^{-1}t}\right)^{\frac{1}{2}}$

$\Rightarrow x=a^{\frac{1}{2}\sin^{-1}t}$ and $y=a^{\frac{1}{2}\cos^{-1}t}$

Consider $x=a^{\frac{1}{2}\sin^{-1}t}$

Taking logarithm on both the sides, we obtain

$$\log x=\frac{1}{2}\sin^{-1}t\log a$$

$$\therefore \frac{1}{x}\cdot\frac{dx}{dt}=\frac{1}{2}\log a\cdot\frac{d}{dt}\left(\sin^{-1}t\right)$$

$$\Rightarrow \frac{dx}{dt}=\frac{x}{2}\log a\cdot\frac{1}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dx}{dt}=\frac{x\log a}{2\sqrt{1-t^2}}$$

Then, consider $y=a^{\frac{1}{2}\cos^{-1}t}$

Taking logarithm on both the sides, we obtain

$$\log y=\frac{1}{2}\cos^{-1}t\log a$$

$$\therefore \frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{2}\log a\cdot\frac{d}{dt}\left(\cos^{-1}t\right)$$

$$\Rightarrow \frac{dy}{dt}=\frac{y\log a}{2}\cdot\left(\frac{-1}{\sqrt{1-t^2}}\right)$$

$$\Rightarrow \frac{dy}{dt}=\frac{-y\log a}{2\sqrt{1-t^2}}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-y\log a}{2\sqrt{1-t^2}}\right)}{\left(\frac{x\log a}{2\sqrt{1-t^2}}\right)} = -\frac{y}{x}.$$

Hence, proved.

Exercise 5.7

Question 1:

Find the second order derivatives of the function.

$x^2 + 3x + 2$

Answer

Let $y = x^2 + 3x + 2$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(3x) + \frac{d}{dx}(2) = 2x + 3 + 0 = 2x + 3$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(2x+3) = \frac{d}{dx}(2x) + \frac{d}{dx}(3) = 2 + 0 = 2$$

Question 2:

Find the second order derivatives of the function.

x^{20}

Answer

Let $y = x^{20}$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{20}\right) = 20x^{19}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(20x^{19}\right) = 20\frac{d}{dx}\left(x^{19}\right) = 20 \cdot 19 \cdot x^{18} = 380x^{18}$$

Question 3:

Find the second order derivatives of the function.

$x \cdot \cos x$

Answer

Let $y = x \cdot \cos x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(x \cdot \cos x) = \cos x \cdot \frac{d}{dx}(x) + x\frac{d}{dx}(\cos x) = \cos x \cdot 1 + x(-\sin x) = \cos x - x\sin x$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}[\cos x - x\sin x] = \frac{d}{dx}(\cos x) - \frac{d}{dx}(x\sin x)$$

$$= -\sin x - \left[\sin x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\sin x)\right]$$

$$= -\sin x - (\sin x + x\cos x)$$

$$= -(x\cos x + 2\sin x)$$

Question 4:

Find the second order derivatives of the function.

$\log x$

Answer

Let $y = \log x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

Question 5:

Find the second order derivatives of the function.

$x^3 \log x$

Answer

Let $y = x^3 \log x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^3 \log x\right] = \log x \cdot \frac{d}{dx}\left(x^3\right) + x^3 \cdot \frac{d}{dx}(\log x)$$

$$= \log x \cdot 3x^2 + x^3 \cdot \frac{1}{x} = \log x \cdot 3x^2 + x^2$$

$$= x^2\left(1 + 3\log x\right)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[x^2\left(1 + 3\log x\right)\right]$$

$$= \left(1 + 3\log x\right) \cdot \frac{d}{dx}\left(x^2\right) + x^2 \frac{d}{dx}\left(1 + 3\log x\right)$$

$$= \left(1 + 3\log x\right) \cdot 2x + x^2 \cdot \frac{3}{x}$$

$$= 2x + 6x\log x + 3x$$

$$= 5x + 6x\log x$$

$$= x\left(5 + 6\log x\right)$$

Question 6:

Find the second order derivatives of the function.

$e^x \sin 5x$

Answer

Let $y = e^x \sin 5x$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x \sin 5x\right) = \sin 5x \cdot \frac{d}{dx}\left(e^x\right) + e^x \frac{d}{dx}(\sin 5x)$$

$$= \sin 5x \cdot e^x + e^x \cdot \cos 5x \cdot \frac{d}{dx}(5x) = e^x \sin 5x + e^x \cos 5x \cdot 5$$

$$= e^x(\sin 5x + 5\cos 5x)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[e^x(\sin 5x + 5\cos 5x)\right]$$

$$= (\sin 5x + 5\cos 5x) \cdot \frac{d}{dx}\left(e^x\right) + e^x \cdot \frac{d}{dx}(\sin 5x + 5\cos 5x)$$

$$= (\sin 5x + 5\cos 5x)e^x + e^x\left[\cos 5x \cdot \frac{d}{dx}(5x) + 5(-\sin 5x) \cdot \frac{d}{dx}(5x)\right]$$

$$= e^x(\sin 5x + 5\cos 5x) + e^x(5\cos 5x - 25\sin 5x)$$

$$= e^x(10\cos 5x - 24\sin 5x) = 2e^x(5\cos 5x - 12\sin 5x)$$

Then,

Question 7:

Find the second order derivatives of the function.

$e^{6x}\cos 3x$

Answer

Let $y = e^{6x}\cos 3x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{6x}\cdot\cos 3x\right) = \cos 3x\cdot\frac{d}{dx}\left(e^{6x}\right) + e^{6x}\cdot\frac{d}{dx}(\cos 3x)$$

$$= \cos 3x\cdot e^{6x}\cdot\frac{d}{dx}(6x) + e^{6x}\cdot(-\sin 3x)\cdot\frac{d}{dx}(3x)$$

$$= 6e^{6x}\cos 3x - 3e^{6x}\sin 3x \qquad \ldots(1)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right) = 6\cdot\frac{d}{dx}\left(e^{6x}\cos 3x\right) - 3\cdot\frac{d}{dx}\left(e^{6x}\sin 3x\right)$$

$$= 6\cdot\left[6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right] - 3\cdot\left[\sin 3x\cdot\frac{d}{dx}\left(e^{6x}\right) + e^{6x}\cdot\frac{d}{dx}(\sin 3x)\right] \qquad [\text{Using } (1)]$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 3\left[\sin 3x\cdot e^{6x}\cdot 6 + e^{6x}\cdot\cos 3x\cdot 3\right]$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 18e^{6x}\sin 3x - 9e^{6x}\cos 3x$$

$$= 27e^{6x}\cos 3x - 36e^{6x}\sin 3x$$

$$= 9e^{6x}\left(3\cos 3x - 4\sin 3x\right)$$

Question 8:

Find the second order derivatives of the function.

$\tan^{-1} x$

Answer

Let $y = \tan^{-1} x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{1+x^2}\right)=\frac{d}{dx}\left(1+x^2\right)^{-1}=(-1)\cdot\left(1+x^2\right)^{-2}\cdot\frac{d}{dx}\left(1+x^2\right)$$

$$=\frac{-1}{\left(1+x^2\right)^2}\times 2x=\frac{-2x}{\left(1+x^2\right)^2}$$

Question 9:

Find the second order derivatives of the function.

$\log(\log x)$

Answer

Let $y=\log(\log x)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log(\log x)\right]=\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)=\frac{1}{x\log x}=(x\log x)^{-1}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[(x\log x)^{-1}\right]=(-1)\cdot(x\log x)^{-2}\cdot\frac{d}{dx}(x\log x)$$

$$=\frac{-1}{(x\log x)^2}\cdot\left[\log x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log x)\right]$$

$$=\frac{-1}{(x\log x)^2}\cdot\left[\log x\cdot 1+x\cdot\frac{1}{x}\right]=\frac{-(1+\log x)}{(x\log x)^2}$$

Question 10:

Find the second order derivatives of the function.

$\sin(\log x)$

Answer

Let $y=\sin(\log x)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin(\log x)\right]=\cos(\log x)\cdot\frac{d}{dx}(\log x)=\frac{\cos(\log x)}{x}$$

$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$

$$=\frac{x\cdot\frac{d}{dx}\left[\cos(\log x)\right]-\cos(\log x)\cdot\frac{d}{dx}(x)}{x^2}$$

$$=\frac{x\cdot\left[-\sin(\log x)\cdot\frac{d}{dx}(\log x)\right]-\cos(\log x).1}{x^2}$$

$$=\frac{-x\sin(\log x).\frac{1}{x}-\cos(\log x)}{x^2}$$

$$=\frac{-\left[\sin(\log x)+\cos(\log x)\right]}{x^2}$$

Question 11:

If $y=5\cos x-3\sin x$, prove that $\frac{d^2y}{dx^2}+y=0$

Answer

It is given that, $y=5\cos x-3\sin x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(5\cos x)-\frac{d}{dx}(3\sin x)=5\frac{d}{dx}(\cos x)-3\frac{d}{dx}(\sin x)$$
$$=5(-\sin x)-3\cos x=-(5\sin x+3\cos x)$$
$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[-(5\sin x+3\cos x)\right]$$
$$=-\left[5\cdot\frac{d}{dx}(\sin x)+3\cdot\frac{d}{dx}(\cos x)\right]$$
$$=-\left[5\cos x+3(-\sin x)\right]$$
$$=-\left[5\cos x-3\sin x\right]$$
$$=-y$$
$$\therefore \frac{d^2y}{dx^2}+y=0$$

Hence, proved.

Question 12:

$y=\cos^{-1}x$, find $\frac{d^2y}{dx^2}$

Answer

It is given that, $y=\cos^{-1}x$

IfThen, in terms of y alone.

$$\frac{dy}{dx}=\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}}=-\left(1-x^2\right)^{\frac{-1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[-\left(1-x^2\right)^{\frac{-1}{2}}\right]$$

$$=-\left(-\frac{1}{2}\right)\cdot\left(1-x^2\right)^{\frac{-3}{2}}\cdot\frac{d}{dx}\left(1-x^2\right)$$

$$=\frac{1}{2\sqrt{\left(1-x^2\right)^3}}\times\left(-2x\right)$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-x}{\sqrt{\left(1-x^2\right)^3}}\qquad\ldots(\text{i})$$

$y=\cos^{-1}x\Rightarrow x=\cos y$

Putting $x=\cos y$ in equation (i), we obtain

$$\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{\left(1-\cos^2 y\right)^3}}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{\left(\sin^2 y\right)^3}}$$

$$=\frac{-\cos y}{\sin^3 y}$$

$$=\frac{-\cos y}{\sin y}\times\frac{1}{\sin^2 y}$$

$$\Rightarrow\frac{d^2y}{dx^2}=-\cot y\cdot\operatorname{cosec}^2 y$$

Question 13:

If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2 y_2 + xy_1 + y = 0$

Answer

It is given that, $y = 3\cos(\log x) + 4\sin(\log x)$

Then,

$$y_1 = 3\cdot\frac{d}{dx}\left[\cos(\log x)\right] + 4\cdot\frac{d}{dx}\left[\sin(\log x)\right]$$

$$= 3\cdot\left[-\sin(\log x)\cdot\frac{d}{dx}(\log x)\right] + 4\cdot\left[\cos(\log x)\cdot\frac{d}{dx}(\log x)\right]$$

$$\therefore y_1 = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x} = \frac{4\cos(\log x) - 3\sin(\log x)}{x}$$

$$\therefore y_2 = \frac{d}{dx}\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right)$$

$$= \frac{x\{4\cos(\log x) - 3\sin(\log x)\}' - \{4\cos(\log x) - 3\sin(\log x)\}(x)'}{x^2}$$

$$= \frac{x\left[4\{\cos(\log x)\}' - 3\{\sin(\log x)\}'\right] - \{4\cos(\log x) - 3\sin(\log x)\}.1}{x^2}$$

$$= \frac{x\left[-4\sin(\log x).(\log x)' - 3\cos(\log x).(\log x)'\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{x\left[-4\sin(\log x).\frac{1}{x} - 3\cos(\log x).\frac{1}{x}\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-4\sin(\log x) - 3\cos(\log x) - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

$$\therefore x^2y_2 + xy_1 + y$$

$$= x^2\left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2}\right) + x\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right) + 3\cos(\log x) + 4\sin(\log x)$$

$$= -\sin(\log x) - 7\cos(\log x) + 4\cos(\log x) - 3\sin(\log x) + 3\cos(\log x) + 4\sin(\log x)$$

$$= 0$$

Hence, proved.

Question 14:

If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Answer

It is given that, $y = Ae^{mx} + Be^{nx}$

Then,

$$\frac{dy}{dx} = A\cdot\frac{d}{dx}\left(e^{mx}\right) + B\cdot\frac{d}{dx}\left(e^{nx}\right) = A\cdot e^{mx}\cdot\frac{d}{dx}(mx) + B\cdot e^{nx}\cdot\frac{d}{dx}(nx) = Ame^{mx} + Bne^{nx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(Ame^{mx} + Bne^{nx}\right) = Am\cdot\frac{d}{dx}\left(e^{mx}\right) + Bn\cdot\frac{d}{dx}\left(e^{nx}\right)$$

$$= Am\cdot e^{mx}\cdot\frac{d}{dx}(mx) + Bn\cdot e^{nx}\cdot\frac{d}{dx}(nx) = Am^2e^{mx} + Bn^2e^{nx}$$

$$\therefore \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$$= Am^2e^{mx} + Bn^2e^{nx} - (m+n)\cdot\left(Ame^{mx} + Bne^{nx}\right) + mn\left(Ae^{mx} + Be^{nx}\right)$$

$$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx}$$

$$= 0$$

Hence, proved.

Question 15:

If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$

Answer

It is given that, $y = 500e^{7x} + 600e^{-7x}$

Then,

$$\frac{dy}{dx} = 500.\frac{d}{dx}\left(e^{7x}\right) + 600.\frac{d}{dx}\left(e^{-7x}\right)$$
$$= 500 \cdot e^{7x} \cdot \frac{d}{dx}(7x) + 600 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x)$$
$$= 3500e^{7x} - 4200e^{-7x}$$
$$\therefore \frac{d^2y}{dx^2} = 3500 \cdot \frac{d}{dx}\left(e^{7x}\right) - 4200.\frac{d}{dx}\left(e^{-7x}\right)$$
$$= 3500 \cdot e^{7x}.\frac{d}{dx}(7x) - 4200 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x)$$
$$= 7 \times 3500 \cdot e^{7x} + 7 \times 4200 \cdot e^{-7x}$$
$$= 49 \times 500e^{7x} + 49 \times 600e^{-7x}$$
$$= 49\left(500e^{7x} + 600e^{-7x}\right)$$
$$= 49y$$

Hence, proved.

Question 16:

If $e^y(x+1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

Answer

The given relationship is $e^y(x+1) = 1$

$$e^{y}(x+1)=1$$

$$\Rightarrow e^{y}=\frac{1}{x+1}$$

Taking logarithm on both the sides, we obtain

$$y=\log\frac{1}{(x+1)}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{dy}{dx}=(x+1)\frac{d}{dx}\left(\frac{1}{x+1}\right)=(x+1).\frac{-1}{(x+1)^{2}}=\frac{-1}{x+1}$$

$$\therefore\frac{d^{2}y}{dx^{2}}=-\frac{d}{dx}\left(\frac{1}{x+1}\right)=-\left(\frac{-1}{(x+1)^{2}}\right)=\frac{1}{(x+1)^{2}}$$

$$\Rightarrow\frac{d^{2}y}{dx^{2}}=\left(\frac{-1}{x+1}\right)^{2}$$

$$\Rightarrow\frac{d^{2}y}{dx^{2}}=\left(\frac{dy}{dx}\right)^{2}$$

Hence, proved.

Question 17:

If $y=\left(\tan^{-1}x\right)^2$, show that $\left(x^2+1\right)^2 y_2+2x\left(x^2+1\right)y_1=2$

Answer

The given relationship is $y=\left(\tan^{-1}x\right)^2$

Then,

$$y_1=2\tan^{-1}x\frac{d}{dx}\left(\tan^{-1}x\right)$$

$$\Rightarrow y_1=2\tan^{-1}x.\frac{1}{1+x^2}$$

$$\Rightarrow \left(1+x^2\right)y_1=2\tan^{-1}x$$

Again differentiating with respect to x on both the sides, we obtain

$$\left(1+x^2\right)y_2+2xy_1=2\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow \left(1+x^2\right)^2 y_2+2x\left(1+x^2\right)y_1=2$$

Hence, proved.

Exercise 5.8

Question 1:

Verify Rolle's Theorem for the function $f(x) = x^2 + 2x - 8,\ x \in [-4, 2]$

Answer

The given function, $f(x) = x^2 + 2x - 8$, being a polynomial function, is continuous in [−4, 2] and is differentiable in (−4, 2).

$$f(-4) = (-4)^2 + 2\times(-4) - 8 = 16 - 8 - 8 = 0$$
$$f(2) = (2)^2 + 2\times 2 - 8 = 4 + 4 - 8 = 0$$

∴ f (−4) = f (2) = 0

⇒ The value of f (x) at −4 and 2 coincides.

Rolle's Theorem states that there is a point c ∈ (−4, 2) such that $f'(c) = 0$

$$f(x) = x^2 + 2x - 8$$
$$\Rightarrow f'(x) = 2x + 2$$
$$\therefore f'(c) = 0$$
$$\Rightarrow 2c + 2 = 0$$
$$\Rightarrow c = -1, \text{ where } c = -1 \in (-4, 2)$$

Hence, Rolle's Theorem is verified for the given function.

Question 2:

Examine if Rolle's Theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's Theorem from these examples?

(i) $f(x)=[x]$ for $x \in [5, 9]$

(ii) $f(x)=[x]$ for $x \in [-2, 2]$

(iii) $f(x)=x^2-1$ for $x \in [1, 2]$

Answer

By Rolle's Theorem, for a function $f:[a, b] \to \mathbf{R}$, if

(a) f is continuous on [a, b]

(b) f is differentiable on (a, b)

(c) f (a) = f (b)

then, there exists some c $\in$ (a, b) such that $f'(c)=0$

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) $f(x)=[x]$ for $x \in [5, 9]$

It is evident that the given function f (x) is not continuous at every integral point.

In particular, f(x) is not continuous at x = 5 and x = 9

⇒ f (x) is not continuous in [5, 9].

Also, $f(5)=[5]=5$ and $f(9)=[9]=9$

$\therefore f(5)\neq f(9)$

The differentiability of f in (5, 9) is checked as follows.
Let n be an integer such that n ∈ (5, 9).

The left hand limit of f at $x=n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x=n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = n are not equal, f is not differentiable at x = n

∴f is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x)=[x]$ for $x\in[5,\ 9]$.

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

It is evident that the given function f (x) is not continuous at every integral point.

In particular, f(x) is not continuous at x = −2 and x = 2

⇒ f (x) is not continuous in [−2, 2].

Also, $f(-2)=[-2]=-2$ and $f(2)=[2]=2$

$\therefore f(-2)\neq f(2)$

The differentiability of f in (−2, 2) is checked as follows.
Let n be an integer such that n ∈ (−2, 2).

The left hand limit of f at $x=n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x=n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = n are not equal, f is not differentiable at x = n

∴f is not differentiable in (−2, 2).

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x)=[x]$ for $x\in[-2,\ 2]$.

(iii) $f(x)=x^2-1$ for $x\in[1,\ 2]$

It is evident that f, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

$$f(1)=(1)^2-1=0$$
$$f(2)=(2)^2-1=3$$

$\therefore$f (1) ≠ f (2)

It is observed that f does not satisfy a condition of the hypothesis of Rolle's Theorem.
Hence, Rolle's Theorem is not applicable for $f(x)=x^2-1$ for $x\in[1,\ 2]$.

Question 3:

If $f:[-5,5]\to\mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5)\neq f(5)$.

Answer

It is given that $f:[-5,5]\to\mathbf{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) f is continuous on [−5, 5].

(b) f is differentiable on (−5, 5).

Therefore, by the Mean Value Theorem, there exists c ∈ (−5, 5) such that

$$f'(c)=\frac{f(5)-f(-5)}{5-(-5)}$$
$$\Rightarrow 10f'(c)=f(5)-f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

$$\therefore f'(c)\neq 0$$
$$\Rightarrow 10f'(c)\neq 0$$
$$\Rightarrow f(5)-f(-5)\neq 0$$
$$\Rightarrow f(5)\neq f(-5)$$

Hence, proved.

Question 4:

Verify Mean Value Theorem, if $f(x)=x^2-4x-3$ in the interval $[a, b]$, where $a=1$ and $b=4$.

Answer

The given function is $f(x)=x^2-4x-3$

f, being a polynomial function, is continuous in [1, 4] and is differentiable in (1, 4) whose derivative is 2x − 4.

$$f(1)=1^2-4\times1-3=-6,\ f(4)=4^2-4\times4-3=-3$$
$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(4)-f(1)}{4-1}=\frac{-3-(-6)}{3}=\frac{3}{3}=1$$

Mean Value Theorem states that there is a point c ∈ (1, 4) such that $f'(c)=1$

$$f'(c)=1$$
$$\Rightarrow 2c-4=1$$
$$\Rightarrow c=\frac{5}{2},\text{ where } c=\frac{5}{2}\in(1,\ 4)$$

Hence, Mean Value Theorem is verified for the given function.

Question 5:

Verify Mean Value Theorem, if $f(x)=x^3-5x^2-3x$

b = 3. Find all $c\in(1,3)$ for which $f'(c)=0$

Answer

The given function f is $f(x)=x^3-5x^2-3x$ in the interval [a, b], where a = 1 and

f, being a polynomial function, is continuous in [1, 3] and is differentiable in (1, 3) whose derivative is $3x^2 - 10x - 3$.

$$f(1)=1^3-5\times1^2-3\times1=-7,\ f(3)=3^3-5\times3^2-3\times3=-27$$
$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(3)-f(1)}{3-1}=\frac{-27-(-7)}{3-1}=-10$$

Mean Value Theorem states that there exist a point c ∈ (1, 3) such that $f'(c)=-10$

$$f'(c)=-10$$
$$\Rightarrow 3c^2-10c-3=10$$
$$\Rightarrow 3c^2-10c+7=0$$
$$\Rightarrow 3c^2-3c-7c+7=0$$
$$\Rightarrow 3c(c-1)-7(c-1)=0$$
$$\Rightarrow (c-1)(3c-7)=0$$
$$\Rightarrow c=1,\ \frac{7}{3},\ \text{where } c=\frac{7}{3}\in(1,\ 3)$$

Hence, Mean Value Theorem is verified for the given function and $c=\frac{7}{3}\in(1,\ 3)$ is the only point for which $f'(c)=0$

Question 6:

Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Answer

Mean Value Theorem states that for a function $f:[a,\ \mathrm{b}]\rightarrow\mathbf{R}$, if

(a) f is continuous on [a, b]

(b) f is differentiable on (a, b)

then, there exists some c $\in$ (a, b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.

$f(x)=[x]$ for $x\in[5,\ 9]$

(i)

It is evident that the given function f (x) is not continuous at every integral point.

In particular, f(x) is not continuous at x = 5 and x = 9

$\Rightarrow$ f (x) is not continuous in [5, 9].

The differentiability of f in (5, 9) is checked as follows.

Let n be an integer such that n $\in$ (5, 9).

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-}\frac{[n+h]-[n]}{h} = \lim_{h\to 0^-}\frac{n-1-n}{h} = \lim_{h\to 0^-}\frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+}\frac{[n+h]-[n]}{h} = \lim_{h\to 0^+}\frac{n-n}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of f at x = n are not equal, f is not differentiable at x = n

∴f is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[5,\ 9]$.

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

It is evident that the given function f (x) is not continuous at every integral point.
In particular, f(x) is not continuous at x = −2 and x = 2

⇒ f (x) is not continuous in [−2, 2].

The differentiability of f in (−2, 2) is checked as follows.

Let n be an integer such that n ∈ (−2, 2).

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = n are not equal, f is not differentiable at x = n

∴f is not differentiable in (−2, 2).

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[-2,\ 2]$.

(iii) $f(x)=x^2-1$ for $x\in[1,\ 2]$

It is evident that f, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

It is observed that f satisfies all the conditions of the hypothesis of Mean Value Theorem.
Hence, Mean Value Theorem is applicable for $f(x)=x^2-1$ for $x\in[1,\ 2]$.

It can be proved as follows.

$f(1)=1^2-1=0,\ f(2)=2^2-1=3$

$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(2)-f(1)}{2-1}=\frac{3-0}{1}=3$

$f'(x)=2x$

$\therefore f'(c)=3$

$\Rightarrow 2c=3$

$\Rightarrow c=\frac{3}{2}=1.5,\text{ where } 1.5\in[1,2]$

Miscellaneous Solutions

Question 1:

$\left(3x^2-9x+5\right)^9$

Answer

Let $y=\left(3x^2-9x+5\right)^9$

Using chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left(3x^2-9x+5\right)^9$$
$$=9\left(3x^2-9x+5\right)^8\cdot\frac{d}{dx}\left(3x^2-9x+5\right)$$
$$=9\left(3x^2-9x+5\right)^8\cdot(6x-9)$$
$$=9\left(3x^2-9x+5\right)^8\cdot3(2x-3)$$
$$=27\left(3x^2-9x+5\right)^8(2x-3)$$

Question 2:

$\sin^3 x + \cos^6 x$

Answer

Let $y = \sin^3 x + \cos^6 x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\sin^3 x\right) + \frac{d}{dx}\left(\cos^6 x\right)$$

$$= 3\sin^2 x \cdot \frac{d}{dx}(\sin x) + 6\cos^5 x \cdot \frac{d}{dx}(\cos x)$$

$$= 3\sin^2 x \cdot \cos x + 6\cos^5 x \cdot (-\sin x)$$

$$= 3\sin x \cos x\left(\sin x - 2\cos^4 x\right)$$

Question 3:

$(5x)^{3\cos 2x}$

Answer

Let $y=(5x)^{3\cos 2x}$

Taking logarithm on both the sides, we obtain

$\log y = 3\cos 2x \log 5x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = 3\left[\log 5x \cdot \frac{d}{dx}(\cos 2x) + \cos 2x \cdot \frac{d}{dx}(\log 5x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[\log 5x(-\sin 2x)\cdot\frac{d}{dx}(2x) + \cos 2x \cdot \frac{1}{5x}\cdot\frac{d}{dx}(5x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[-2\sin 2x \log 5x + \frac{\cos 2x}{x}\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$$

$$\therefore \frac{dy}{dx} = (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$$

Question 4:

$\sin^{-1}\left(x\sqrt{x}\right),\ 0 \le x \le 1$

Answer

Let $y = \sin^{-1}\left(x\sqrt{x}\right)$

Using chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\sin^{-1}\left(x\sqrt{x}\right)$$
$$=\frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^2}}\times\frac{d}{dx}\left(x\sqrt{x}\right)$$
$$=\frac{1}{\sqrt{1-x^3}}\cdot\frac{d}{dx}\left(x^{\frac{3}{2}}\right)$$
$$=\frac{1}{\sqrt{1-x^3}}\times\frac{3}{2}\cdot x^{\frac{1}{2}}$$
$$=\frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$
$$=\frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

Question 5:

$$\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}},\ -2<x<2$$

Answer

Let $y = \dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$

By quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\sqrt{2x+7}\frac{d}{dx}\left(\cos^{-1}\frac{x}{2}\right) - \left(\cos^{-1}\frac{x}{2}\right)\frac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^2}$$

$$= \frac{\sqrt{2x+7}\left[\dfrac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\dfrac{d}{dx}\left(\dfrac{x}{2}\right)\right] - \left(\cos^{-1}\frac{x}{2}\right)\dfrac{1}{2\sqrt{2x+7}}\cdot\dfrac{d}{dx}(2x+7)}{2x+7}$$

$$= \frac{\sqrt{2x+7}\dfrac{-1}{\sqrt{4-x^2}} - \left(\cos^{-1}\frac{x}{2}\right)\dfrac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$= \frac{-\sqrt{2x+7}}{\sqrt{4-x^2}\times(2x+7)} - \frac{\cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

$$= -\left[\frac{1}{\sqrt{4-x^2}\sqrt{2x+7}} + \frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

Question 6:

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0 < x < \frac{\pi}{2}$$

Answer

Let $y=\cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$ ...(1)

Then, $\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

$$=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}$$

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1-\sin x)(1+\sin x)}}{(1+\sin x)-(1-\sin x)}$$

$$=\frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}$$

$$=\frac{1+\cos x}{\sin x}$$

$$=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\cot\frac{x}{2}$$

Therefore, equation (1) becomes

$$y=\cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow y=\frac{x}{2}$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}(x)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}$$

Question 7:

$(\log x)^{\log x}$, $x > 1$

Answer

Let $y = (\log x)^{\log x}$

Taking logarithm on both the sides, we obtain

$$\log y = \log x \cdot \log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[\log x \cdot \log(\log x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\log x).\frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log(\log x).\frac{1}{x} + \log x \cdot \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x}\log(\log x) + \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$$

Question 8:

$\cos(a\cos x + b\sin x)$, for some constant a and b.

Answer

Let $y = \cos(a\cos x + b\sin x)$

By using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\cos(a\cos x + b\sin x)$$

$$\Rightarrow \frac{dy}{dx} = -\sin(a\cos x + b\sin x)\cdot\frac{d}{dx}(a\cos x + b\sin x)$$

$$= -\sin(a\cos x + b\sin x)\cdot\left[a(-\sin x) + b\cos x\right]$$

$$= (a\sin x - b\cos x)\cdot\sin(a\cos x + b\sin x)$$

Question 9:

$(\sin x - \cos x)^{(\sin x - \cos x)}$, $\frac{\pi}{4} < x < \frac{3\pi}{4}$

Answer

Let $y = (\sin x - \cos x)^{(\sin x - \cos x)}$

Taking logarithm on both the sides, we obtain

$$\log y = \log\left[(\sin x - \cos x)^{(\sin x - \cos x)}\right]$$

$$\Rightarrow \log y = (\sin x - \cos x) \cdot \log(\sin x - \cos x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[(\sin x - \cos x)\log(\sin x - \cos x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x) \cdot \frac{d}{dx}(\sin x - \cos x) + (\sin x - \cos x) \cdot \frac{d}{dx}\log(\sin x - \cos x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x) \cdot (\cos x + \sin x) + (\sin x - \cos x) \cdot \frac{1}{(\sin x - \cos x)} \cdot \frac{d}{dx}(\sin x - \cos x)$$

$$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}\left[(\cos x + \sin x) \cdot \log(\sin x - \cos x) + (\cos x + \sin x)\right]$$

$$\therefore \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}(\cos x + \sin x)\left[1 + \log(\sin x - \cos x)\right]$$

Question 10:

$x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$

Answer

Let $y = x^x + x^a + a^x + a^a$

Also, let $x^x = u,\ x^a = v,\ a^x = w,$ and $a^a = s$

$\therefore\ y = u + v + w + s$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx} \qquad ...(1)$$

$u = x^x$

$\Rightarrow \log u = \log x^x$

$\Rightarrow \log u = x \log x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x \cdot 1 + x \cdot \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^x[\log x + 1] = x^x(1 + \log x) \qquad ...(2)$$

$$v = x^a$$

$$\therefore \frac{dv}{dx} = \frac{d}{dx}(x^a)$$

$$\Rightarrow \frac{dv}{dx} = ax^{a-1} \qquad ...(3)$$

$$w = a^x$$

$$\Rightarrow \log w = \log a^x$$

$$\Rightarrow \log w = x \log a$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{w} \cdot \frac{dw}{dx} = \log a \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \frac{dw}{dx} = w \log a$$

$$\Rightarrow \frac{dw}{dx} = a^x \log a \qquad ...(4)$$

$s = a^a$

Since a is constant, a^a is also a constant.

$$\therefore \frac{ds}{dx} = 0 \qquad ...(5)$$

From (1), (2), (3), (4), and (5), we obtain

$$\frac{dy}{dx} = x^x\left(1+\log x\right) + ax^{a-1} + a^x \log a + 0$$
$$= x^x\left(1+\log x\right) + ax^{a-1} + a^x \log a$$

Question 11:

$x^{x^2-3}+(x-3)^{x^2}$, for $x>3$

Answer

Let $y=x^{x^2-3}+(x-3)^{x^2}$

Also, let $u=x^{x^2-3}$ and $v=(x-3)^{x^2}$

$\therefore y=u+v$

Differentiating both sides with respect to x, we obtain

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$u=x^{x^2-3}$

$\therefore \log u=\log\left(x^{x^2-3}\right)$

$\log u=\left(x^2-3\right)\log x$

Differentiating with respect to x, we obtain

$$\frac{1}{u}\cdot\frac{du}{dx}=\log x\cdot\frac{d}{dx}\left(x^2-3\right)+\left(x^2-3\right)\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=\log x\cdot 2x+\left(x^2-3\right)\cdot\frac{1}{x}$$

$$\Rightarrow \frac{du}{dx}=x^{x^2-3}\cdot\left[\frac{x^2-3}{x}+2x\log x\right]$$

Also,

$v=(x-3)^{x^2}$

$\therefore \log v=\log(x-3)^{x^2}$

$\Rightarrow \log v=x^2\log(x-3)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\log(x-3)\cdot\frac{d}{dx}(x^2)+x^2\cdot\frac{d}{dx}[\log(x-3)]$$
$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\log(x-3)\cdot 2x+x^2\cdot\frac{1}{x-3}\cdot\frac{d}{dx}(x-3)$$
$$\Rightarrow\frac{dv}{dx}=v\left[2x\log(x-3)+\frac{x^2}{x-3}\cdot 1\right]$$
$$\Rightarrow\frac{dv}{dx}=(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$$

Substituting the expressions of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (1), we obtain

$$\frac{dy}{dx}=x^{x^2-3}\left[\frac{x^2-3}{x}+2x\log x\right]+(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$$

Question 12:

Find $\frac{dy}{dx}$, if $y=12(1-\cos t), x=10(t-\sin t), -\frac{\pi}{2}<t<\frac{\pi}{2}$

Answer

It is given that, $y=12(1-\cos t), x=10(t-\sin t)$

$$\therefore\frac{dx}{dt}=\frac{d}{dt}[10(t-\sin t)]=10\cdot\frac{d}{dt}(t-\sin t)=10(1-\cos t)$$
$$\frac{dy}{dt}=\frac{d}{dt}[12(1-\cos t)]=12\cdot\frac{d}{dt}(1-\cos t)=12\cdot[0-(-\sin t)]=12\sin t$$
$$\therefore\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{12\sin t}{10(1-\cos t)}=\frac{12\cdot 2\sin\frac{t}{2}\cdot\cos\frac{t}{2}}{10\cdot 2\sin^2\frac{t}{2}}=\frac{6}{5}\cot\frac{t}{2}$$

Question 13:

Find $\frac{dy}{dx}$, if $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2},\ -1\le x\le 1$

Answer

It is given that, $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left[\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(\sin^{-1}x\right)+\frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\cdot\frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{x}\cdot\frac{1}{2\sqrt{1-x^2}}\cdot\frac{d}{dx}\left(1-x^2\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{2x\sqrt{1-x^2}}(-2x)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx}=0$$

Question 14:

If $x\sqrt{1+y}+y\sqrt{1+x}=0$

$$\frac{dy}{dx}=-\frac{1}{(1+x)^2}$$

Answer , for, −1 < x <1, prove that

It is given that,

$$x\sqrt{1+y}+y\sqrt{1+x}=0$$

$$\Rightarrow x\sqrt{1+y}=-y\sqrt{1+x}$$

Squaring both sides, we obtain

$$x^2(1+y)=y^2(1+x)$$

$$\Rightarrow x^2+x^2y=y^2+xy^2$$

$$\Rightarrow x^2-y^2=xy^2-x^2y$$

$$\Rightarrow x^2-y^2=xy(y-x)$$

$$\Rightarrow (x+y)(x-y)=xy(y-x)$$

$$\therefore x+y=-xy$$

$$\Rightarrow (1+x)y=-x$$

$$\Rightarrow y=\frac{-x}{(1+x)}$$

Differentiating both sides with respect to x, we obtain

$$y=\frac{-x}{(1+x)}$$

$$\frac{dy}{dx}=-\frac{(1+x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+x)}{(1+x)^2}=-\frac{(1+x)-x}{(1+x)^2}=-\frac{1}{(1+x)^2}$$

Hence,

Question 15:

If $(x-a)^2+(y-b)^2=c^2$, for some $c>0$, prove that $\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$ is a constant independent of a and b

$$\frac{d}{dx}\left[(x-a)^2\right]+\frac{d}{dx}\left[(y-b)^2\right]=\frac{d}{dx}\left(c^2\right)$$

$$\Rightarrow 2(x-a)\cdot\frac{d}{dx}(x-a)+2(y-b)\cdot\frac{d}{dx}(y-b)=0$$

$$\Rightarrow 2(x-a)\cdot 1+2(y-b)\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-a)}{y-b} \quad ...(1)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{-(x-a)}{y-b}\right]$$

Answer

It is given that, $(x-a)^2+(y-b)^2=c^2$

Differentiating both sides with respect to x, we obtain

$$= -\left[\frac{(y-b)\cdot\frac{d}{dx}(x-a)-(x-a)\cdot\frac{d}{dx}(y-b)}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)-(x-a)\cdot\frac{dy}{dx}}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)-(x-a)\cdot\left\{\frac{-(x-a)}{y-b}\right\}}{(y-b)^2}\right] \qquad [\text{Using (1)}]$$

$$= -\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]$$

$$\therefore \left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^{\frac{3}{2}} = \frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]} = \frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}$$

$$= \frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}} = \frac{\frac{c^3}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}}$$

$= -c$, which is constant and is independent of a and b

Hence, proved.

Question 16:

If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\sin a}$

Answer

It is given that, $\cos y = x\cos(a+y)$

$$\therefore \frac{d}{dx}[\cos y] = \frac{d}{dx}[x\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y)\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}[\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y) + x\cdot[-\sin(a+y)]\frac{dy}{dx}$$

$$\Rightarrow [x\sin(a+y) - \sin y]\frac{dy}{dx} = \cos(a+y) \quad ...(1)$$

Since $\cos y = x\cos(a+y)$, $x = \dfrac{\cos y}{\cos(a+y)}$

Then, equation (1) reduces to

$$\left[\frac{\cos y}{\cos(a+y)}\cdot\sin(a+y) - \sin y\right]\frac{dy}{dx} = \cos(a+y)$$

$$\Rightarrow [\cos y\cdot\sin(a+y) - \sin y\cdot\cos(a+y)]\cdot\frac{dy}{dx} = \cos^2(a+y)$$

$$\Rightarrow \sin(a+y-y)\frac{dy}{dx} = \cos^2(a+b)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+b)}{\sin a}$$

Hence, proved.

Question 17:

If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\dfrac{d^2y}{dx^2}$

Answer

It is given that, $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

$$\therefore \frac{dx}{dt} = a\cdot\frac{d}{dt}(\cos t + t\sin t)$$

$$= a\left[-\sin t + \sin t\cdot\frac{d}{dx}(t) + t\cdot\frac{d}{dt}(\sin t)\right]$$

$$= a[-\sin t + \sin t + t\cos t] = at\cos t$$

$$\frac{dy}{dt} = a\cdot\frac{d}{dt}(\sin t - t\cos t)$$

$$= a\left[\cos t - \left\{\cos t\cdot\frac{d}{dt}(t) + t\cdot\frac{d}{dt}(\cos t)\right\}\right]$$

$$= a\left[\cos t - \{\cos t - t\sin t\}\right] = at\sin t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{at\sin t}{at\cos t} = \tan t$$

Then, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan t) = \sec^2 t\cdot\frac{dt}{dx}$

$$= \sec^2 t\cdot\frac{1}{at\cos t} \qquad \left[\frac{dx}{dt} = at\cos t \Rightarrow \frac{dt}{dx} = \frac{1}{at\cos t}\right]$$

$$= \frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}$$

Question 18:

$f(x) = |x|^3$

If , show that $f''(x)$ exists for all real x, and find it.

Answer

It is known that, $|x|=\begin{cases}x, & \text{if } x\geq 0\\ -x, & \text{if } x<0\end{cases}$

Therefore, when x ≥ 0, $f(x)=|x|^3=x^3$

In this case, $f'(x)=3x^2$ and hence, $f''(x)=6x$

When x < 0, $f(x)=|x|^3=(-x)^3=-x^3$

In this case, $f'(x)=-3x^2$ and hence, $f''(x)=-6x$

Thus, for $f(x)=|x|^3$, $f''(x)$ exists for all real x and is given by,

$f''(x)=\begin{cases}6x, & \text{if } x\geq 0\\ -6x, & \text{if } x<0\end{cases}$

Question 19:

Using mathematical induction prove that $\frac{d}{dx}(x^n)=nx^{n-1}$ for all positive integers n.

Answer

To prove: $P(n):\frac{d}{dx}(x^n)=nx^{n-1}$ for all positive integers n

For n = 1,

$P(1):\frac{d}{dx}(x)=1=1\cdot x^{1-1}$

∴P(n) is true for n = 1

Let P(k) is true for some positive integer k.

That is, $P(k): \frac{d}{dx}\left(x^k\right) = kx^{k-1}$

It has to be proved that P(k + 1) is also true.

Consider $\frac{d}{dx}\left(x^{k+1}\right) = \frac{d}{dx}\left(x \cdot x^k\right)$

$$= x^k \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}\left(x^k\right) \qquad \text{[By applying product rule]}$$

$$= x^k \cdot 1 + x \cdot k \cdot x^{k-1}$$

$$= x^k + kx^k$$

$$= (k+1) \cdot x^k$$

$$= (k+1) \cdot x^{(k+1)-1}$$

Thus, P(k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, the statement P(n) is true for every positive integer n.

Hence, proved.

Question 20:

Using the fact that sin (A + B) = sin A cos B + cos A sin B and the differentiation, obtain the sum formula for cosines.

Answer

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[\sin(A+B)\right] = \frac{d}{dx}(\sin A\cos B) + \frac{d}{dx}(\cos A\sin B)$$

$$\Rightarrow \cos(A+B)\cdot\frac{d}{dx}(A+B) = \cos B\cdot\frac{d}{dx}(\sin A) + \sin A\cdot\frac{d}{dx}(\cos B) + \sin B\cdot\frac{d}{dx}(\cos A) + \cos A\cdot\frac{d}{dx}(\sin B)$$

$$\Rightarrow \cos(A+B)\cdot\frac{d}{dx}(A+B) = \cos B\cdot\cos A\frac{dA}{dx} + \sin A(-\sin B)\frac{dB}{dx} + \sin B(-\sin A)\cdot\frac{dA}{dx} + \cos A\cos B\frac{dB}{dx}$$

$$\Rightarrow \cos(A+B)\cdot\left[\frac{dA}{dx} + \frac{dB}{dx}\right] = (\cos A\cos B - \sin A\sin B)\cdot\left[\frac{dA}{dx} + \frac{dB}{dx}\right]$$

$$\therefore \cos(A+B) = \cos A\cos B - \sin A\sin B$$

Question 22:

If, $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$ prove that $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Answer

$$y=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

$$\Rightarrow y=(mc-nb)f(x)-(lc-na)g(x)+(lb-ma)h(x)$$

Then, $\frac{dy}{dx}=\frac{d}{dx}\left[(mc-nb)f(x)\right]-\frac{d}{dx}\left[(lc-na)g(x)\right]+\frac{d}{dx}\left[(lb-ma)h(x)\right]$

$$=(mc-nb)f'(x)-(lc-na)g'(x)+(lb-ma)h'(x)$$

$$=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Thus, $\frac{dy}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Question 23:

$y=e^{a\cos^{-1}x},-1\le x\le 1$

Answer

IfIt is given that, $y=e^{a\cos^{-1}x}$, show that $\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$

Taking logarithm on both the sides, we obtain

$\log y = a\cos^{-1} x \log e$

$\log y = a\cos^{-1} x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = a \times \frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}}$$

By squaring both the sides, we obtain

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$$

$$\Rightarrow \left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

$$\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Again differentiating both sides with respect to x, we obtain

$$\left(\frac{dy}{dx}\right)^2 \frac{d}{dx}\left(1-x^2\right) + \left(1-x^2\right) \times \frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right] = a^2 \frac{d}{dx}\left(y^2\right)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 (-2x) + \left(1-x^2\right) \times 2\frac{dy}{dx}.\frac{d^2y}{dx^2} = a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 (-2x) + \left(1-x^2\right) \times 2\frac{dy}{dx}.\frac{d^2y}{dx^2} = a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow -x\frac{dy}{dx} + \left(1-x^2\right)\frac{d^2y}{dx^2} = a^2.y \qquad \left[\frac{dy}{dx} \neq 0\right]$$

$$\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

Hence, proved.